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A STUDY OF SOME STOCHASTIC MODELS ON FERTILITY ANALYSIS

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DECLARATION

This thesis entitled 'A study of Some Stochastic Models on Fertility Analysis' that is being submitted by me at the Bundelkhand University, Jhansi for the award of the degree of Doctor of Philosophy is based on my research work carried out under the supervision of Dr. Vijay Kumar Sehgal, Reader, Department of Mathematics and Statistics, Bundelkhand University. This work either in part or in full, has not been submitted to any University or Institution for the award of any degree.

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CHAPTER-I

A STUDY OF SOME STOCHASTIC MODELS ON FERTILITY ANALYSIS

INTRODUCTION

1.1 Fertility analysis is a very important item of investigation in population studies. It refers to the additive component of the growth of population.

Stochastic modelling, of late, has been proved to be a very useful technique for fertility investigation. This is because several socio-economic and biological variables affecting the fertility pattern are either unobservables or their measurements are imprecise due to non-sampling errors. For example, the estimation of proportion of biologically sterile females in a population is beset with many problems baffling the exact estimation of its size. Proportion of mothers without children cannot be considered as the relevant data for the estimation because there are females who remain childless without being sterile under the constraints of physical, social, personal and psychological factors. Under the situation, a judicious use of statistical models for the description of the process and estimation of the relevant parameters often has been found to be indispensable.

Fertility analysis by stochastic models, of late, has drawn considerable attention of mathematically oriented

Demographers; especially because of certain special advantages of the same over the traditional deterministic and actuarial approaches. One of the major advantages of stochastic models in fertility analysis is to derive small sample estimates regarding various fertility parameters with exact specification of sampling errors. Also to evaluate the status of various possible estimators of the Demographic parameters for their relative acceptability under the given set up of a population. In fact, the same not generally available necessary in actuarial approaches as the latter has the advantage of a large number of case records. Besides this, certain demographic parameters are basically unobservables but playing very important role while characterizing qualitatively and quantitatively the structure of a population. Let us take the example of estimating the parameter concerning the proportion of Biologically sterile couples in a population. It may be noted that Biologically sterile couples are unidentifiable from the traditional data of sample surveys. Because the usual way to identity them viz. couples who did not have any children during the whole reproductive (or Marital) span by, no means, provide any clue relating to the size of the Biologically, sterile couples. A sizeable number of couples do appear in records who did not have any surviving children although biologically not sterile. Besides this, there are a number of couples (especially among upper social class in Western Countries) who intend

to remain childless although not being biologically sterile. The estimation of the proportion of biologically sterile couples, is thus not feasible without appropriate stochastic modelling. Second example relating to the effecacies of stochastic modelling in fertility analysis arises in the case of estimation of Post Partum Amenorrhoea. Estimation of post partum amenorrhoea (P.P.A) is beset with many practical difficulties (Biswas (1963), Biswas (1967)). Two different types of non-sampling errors do vitiate the estimation of P.P.A. and these two errors operate in reverse directions in under-estimating and overestimating respectively the P.P.A. Besides that, there exists heterogeneity in the data of P.P.A. which makes the estimation of P.P.A. still difficult ; as the modelling of the P.P.A reveals that the same conforms to a mixture of two independent random variables (Yadava (1966), Biswas (1971), Singh and Bhaduri (1971)). A practical way out to this difficulty using stochastic modelling approach is to assume certain plausible probability distribution for describing the fecundable period, gestation period and the P.P.A. (Talwar (1966)). If the distributions are independent the convolutions of the three would be the distribution between two consecutive order of conceptions (or births) on the assumption of one to one correspondence between the conception and the birth. Hence the synthesized (convoluted) model may be fitted

in the data corresponding to the waiting time between two consecutive births (or conceptions). The reporting of which is much less errorneous than that of the distribution of the post partum amenorrhoea. The fitted data in the model provide estimates of the parameters in the synthesized distribution of which one or two may be pertinent to the period of amenorrhoea. The example illustrates the applicability of stochastic modelling of estimating the P.P.A. without drawing any data relating to the same. However, on estimation of the parameters of the period of post partum amenorrhoea one may fit the theoretical distribution of the P.P.A again in the empirical data to get an idea of the different sorts of response error in the data of P.P.A.

There are many other examples illustrating the indispensabilities of stochastic modelling in the estimation of certain Demographic Paramters. For example, in a family planning action research programme it may be useful to estimate the proportion of female population at a time who are exposed to the risk of conception (fecundable period) and also what is the extent of the same population under infecundable condition (Biswas (1975)) for distribution of family limitation packages and to work out the cost of the same; which is impossible without appropriate stochastic modelling. Similar is the case of estimating the number of births averted under a given family planning action research programme andthe cost benefit analysis of the same.

Finally, apart from exhibiting the applicability of stochastic modelling and almost indispensibility of the same in relation to certain problems it is worthwhile to mention the vital role of stochastic modelling in analysing incomplete demographic data based on follow up studies. Either the reference period of follow up study is fixed in which case the proportion of observations having the characteristic under study is a random variable; or when follow up of the survey is terminated only on collecting a fixed desirable number of cases having a given characteristic the period of termination is itself random variable. Thus we have censored samples of type I and type II respectively. The analysis of incomplete data can only be done by appropriate stochastic modelling of censored distributions.

In what follows the motivations of the models were thus intended to develop Mathematically oriented Demographic research and to support the Biomedical research concerning the biological factors affecting fertility, maternal and child health. While in India emergence of National problem of family limitation (because of rapid decline in the mortality rate with almost unchanging status of fertility condition), no doubt, gave great impetus in the development of probability models concerning human reproductive process; to evolve optimal strategies for the reduction of level of fertility on a natural scale but realistic models befitting the assumption

which the population really conforms remain a tough exercise because of basically two factors viz. (1) the lack of adequate data base to provide the necessary ingredients for appropriate development of the model, (2) Mathematical exercise is discouraged with the replacement of probability models by simulation models presumably due to the ability of the latter type of modelling to take into account of several plausible factors which as an exercise with Probability models may still be a chaliengìng problem. Nevertheless simulation models does not help much in defining a theoretical basis between the explanatory and depending variables in the process.

1.2 A historical survey of the stochastic fertility models and the motivation of this study.

The first systematic study of modelling fertility analysis perhaps started from the paper of Gini (1924) which was presented at the International Mathematics Congress, Toranto where the notation of Fecundable was first introduced. In recent decades there has been considerable amount of interest generated in the analysis of pregnancies of cohorts of married women. Special emphasis of the pattern of live births, is one of the factors studied by Henry (1953). His work (1953,1957,1961) on models on Human Reproductive Process comprised of both discrete and continuous model postulating both homogeneity and heterogeneity of the population.

A historical review of the models that they differ only

in treatment of time as "discrete" or "continuous". In many of the models the probability of conception is held constant and further assumption is made about all conceptions leading to live births which are associated with a fixed non-susceptible period covering the gestation period and the period of post-partum amenorrhoea. However, new type of more flexible and realistic probability models have also been evolved by introducing the notion of time when a conception is recorded. Emphasis is laid on the results for the probability of a recording at a specified time "t" in the same way as it is held in physical system as Geiger Muller Counter Model of Type I and Type II.

The Human Reproductive Process is thus compared with Geiger Muller (G.M) Counter where every event occuring with an intensity is followed by a dead time (or blocked time) during which no further success is possible. The term "counter is paralysed" is used in this respect which is comparable to the non-susceptible period following a pregnancy in the human reproductive process. The G.M Counter is classified according as the dead time is fixed or random variable. Type II G.M Counters are contrasted from type I in the sense that not only there cannot be any success during the dead time in type II (as in type I) but a result of arrival of a particle in G.M. Counter type II during the dead time further prolongs the dead time without registering the event during the blocked period.

For the analysis of human reproductive process, counter model I with fixed or random variable dead time appears to be more pertinent. It will be shown in the subsequent part of the theory that process (e.g. Dandekar's model,etc.) are directly derivable from the existing results of G.M. Counter Models. Further, the renewal theory technique is amenable for the development of probability models on fertility and the human reproductive process. All these models can be classified into two categories. The first deals with the waiting time distribution between two consecutive events (such as births, pregnancies, abortions, etc.), while the second deals with the probability distribution of the number of events during a fixed time.

In the first case, time is a random variable and the system conforms to ordinary or delayed Renewal Process, which is, in general, Non-Markovian in character. In the second case, the number of events during a fixed time is a random variable and conforms to a renewal counting process. It is obvious to find out the former is continuous whereas the latter is discrete and one can directly be obtained from the other.

In India, the formulation of the stochastic model representing the human reproductive process was first done by Dandekar (1955). Dandekar assumed a one to one correspondence between pregnancy and live birth and a constant time independent Poisson intensity for births/pregnancies of all others.

A fixed non-susceptible period (dead time) was assumed following every pregnancy/birth in the pattern of Counter Model Type I. The probability distribution of the number of events in fixed marital duration (0,t] has been obtained from the Modified Poisson Distribution as evolved by Dendekar. However, it may be seen that the distribution considered by Dendekar is nothing but a Poisson Renewal Process where every renewal is followed by a dead time of fixed measure. Thus Dandekar's distribution is otherwise obtainable as an example of Counter Model Type I with fixed dead time as considered by Takacs (1960), Dharmadhikari (1964), Biswas (1973). Further the difference equation techniques adopted by Basu (1955) provides alternative derivation of asymptotic pregnancy rate which is otherwise obtainable from Dandekar's Model.

Brass (1958) obtained Negative Binomial model from Dandekar's distribution ignoring the non-susceptible exposures following every birth and obtained a theoretical model for the probability distribution of births. However, the model did not give good fit in empirical data as infecundable exposure following every birth was ignored. The probability of zero number of births to mothers with completed reproductive span as obtainable from William Brass Model did not show good correspondence with that of the empirical frequency for the same class because of presence of considerable proportion of Biologically sterile mothers (or couples) as well as a

good proportion of mothers (or couples) who wanted to remain effectively childless without being biologically sterile. As William Brass Model made no provision for the same, Brass (1958) while attempting to improve the fit used a "zero truncated" Negative Binomial distribution which implied considerable loss of information. Hence the Brass Model needed further generalisation.

Biswas (1973) made an attempt to generalise Brass Model by removing some of the point of deficiencies of William Brass Model is (i) ignoring the non-susceptible exposure following pregnancy, (ii) ignoring the biologically sterile couples which got mixed up with fecund couples in the population while accounting the distribution of live births.

Dharmadhikari (1964) while attempting to generalise Dandekar's model, first of all developed a model based on Mixture of Exponential distribution with the origin shifted by K (the length of the non-susceptible exposure). Besides this generalisation of Dandekar's model modified into a mixture of two distributions, Dharmadhikari (1964) developed a generalised related time dependent process (Doubly Stochastic Poisson Process) and considered the application of the same in further models.

Singh (1963) independently obtained two different interrupted probability models on similar assumption with the problem of estimating the number of schools of fishes in a fishing ground. For example, the dead time "h" following each conception was comparable to the time for which further

scouting of fishes was temporarily abandoned following the sight of a school of fish and exactly "h" hours were spent for fishing the school. In both the models Singh assumed the presence of $(1-\lambda)$ of the couples to be effectively sterile. While in the first model Singh assumed the monthly probability of conception to be constant, in the second model he assumed the same admitting a probability distribution of Beta type following Potter and Parker. S.N.Singh (1964) further obtained the interruped probability models for couple fertility under the assumption of the presence of a fixed proportion of effectively sterile couple as well as fixed non-susceptible exposure following every conception and obtained BAN estimators of the parameters of the model by using Central Limit Theorem. Modified Minimum χ^2 method has been applied to estimate BAN estimators. As noted by Sheps and Menkin (1969), Singh's model is a modified restatement of Dandekar's model using a Neyman's Method (1949).

Another aspect of S.N. Singh's work in this direction comprises of analysing the building up appropriate probability model for explaining the random changes causing variability in the number of conceptions during a fixed fecundable period. In the first course of his study Singh (1963) devoted to his analysis on complete conception only and obtained BAN estimates of the parameters of the model, whereas in a generalised model built up later, Singh and Bhattacharya (1970) admitted incomplete conceptions too in the model which led to several

more parameters of the model. A partial estimation of some of the parameters was made while assuming the knowledge relating to the rest.

Perrin and Sheps (1964) made an explicit use of Renewal Theory for building of probability models on human reproductive process that allow a variety of pregnancy outcomes (live birth, still birth, foetal wastage, sponteneous or induced abortions) and general distribution for durations of pregnancy and the post-partum non-susceptible status.

The states of the reproductive process in Sheps and Perrin consist of :

S_0 = Non-pregnant fecundable state

S_1 = Pregnant state

S_2 = Post-partum infecundable state associated with abortion or early foetal loss

S_3 = Post-partum infecundable state associated with still birth.

S_4 = Post-partum infecundable state associated with live birth.

The length of stay in each state is viewed as a random variable, the waiting between two renewals of the same state and the number of renewals of each state during a fixed term (0,t] conforms to a Semi Markov pattern. To study the fertility process, it is, important to know the number of passages into a specified state, the fertility rate (or the pregnancy rate per unit of time); and the

probability distribution of the renewal of different states of the process in a given period of time.

The results obtained from these models have implication both for the analysis of data on the distribution of births to a group of women in a calendar period and for the planning and evaluation of family planning programmes. In this direction a family building model has been set up by Sheps, Menkin and Radick (1969). A very important model for studying the short term effects of change in reproductive behaviour (Sheps and Menkin (1971)) on intrinsic birth rate and a model for conceptive delays in a heterogeneous group have also come up.

Basically Stochastic Models relating to fertility analysis can be broadly classified into two categories; viz. (1) Models which predict the number of events (a discrete random variable like births, pregnancies, etc.) during a fixed period of time say (0,t].

(ii) Models which provide the waiting time distribution between two consecutive events or between any two events say ith and jth event (j > i).

One interesting exercise relating to the models of the first type is to consider the total number of conceptions or births during the entire marital span of a particular marriage cohort and compare the same with another proximate marriage cohort (differing by few years) having the same marital age while other socio-economic factors remaining

more or less the same (Chapter-3). The motivation in such an exercise is to find out whether there is a change in the total fertility level of two marriage cohorts. Since the problem relates to basically two marriage cohorts where dates of marriage of the first and the second cohort are assumed to have taken place at two points of time with certain time lag; naturally the technique of Stochastic Point Progress was required to be adopted. The same type of problem has been solved by an entirely new technique known as "Palm Probability" which was hitherto used in "Queueing Processes" (Khintchine (1966)) while facing the second type of problem on Stochastic Models on Fertility Analysis "Palm Probability" also happened to be useful to provide innovative solutions to some problems of dependent processes arising on account of the individual variability of the fecundability parameter following certain probability laws. Stochastically the variation of the fecundability parameter thus leads to a complete destruction of the renewal process, where each renewal corresponds to a conception according to the order of the renewal.

The variability of the parameter leads to the result of interarrival time distribution between different order of conceptions which are correlated. Thus the waiting time distribution between 1st and 2nd order conceptions cannot be treated independently of the waiting time distribution of the 0th (date of effective marriage) and 1st order of

conceptions. Hence it has been envisaged in our exercise to employ the technique of Palm Probability to obtain the conditional waiting time distribution between the first and the second (or higher) order of conceptions given that the first has taken place at a particular time say T=t. The theoretical development concerning this kind of exercise has been given in Chapter-2 and Chapter-4 of this thesis. In the next place, a Multistate Markov Chain model has been developed to study effectively the efficacy of a particular sterilization policy motivated to reduce the birth rate in a population subject to the cultural acceptability of the same in a population (Chapter-5). A further extension of the problem of sterilization of mothers based on number of arising a family planning programme using the approach of Martingales is taken up in Chapter-6.

In Chapter-7 of the thesis a partial solution of two other problems of related interest in the fertility analysis has also been encountered. The first comprises of an examination into the impact of certain family planning action research programme. Suppose that with the onset of a certain order of conception the level of fecundity of a population is reduced; then the problem comprises of observing the impact of the same reduction in the successive spacing between consecutive order of conceptions as well as to measure the reduction in the average number of conceptions following the implementation of the scheme.

The second problem comprises of re-examining the efficacy of the traditional post-partum abstinence as a social custom while reducing the conception rate as an indirect measure. In fact, the period of abstinence is carried on mostly during the period when there is otherwise a post partum infecundable period due to lactational amenorrhoea. Therefore, the motivation in this study is to obtain the probability that the period of post - partum abstinence exceeding that of P.P.A. by a specific non-negative quantity. In fact it is worth-while to consider the difference of the distribution of the period of abstinence and that of post-partum amenorrhoea which may highlight the role of effectiveness of post-partum abstinence.

The chapter-8 of the thesis again employs the method of Palm Probability for obtaining a generalised probability model for measuring inter conception delays.

The chapter-9 deals with a new application of survival analysis technique for obtaining inter live birth intervals by employing a Hazard model which can take into account of any type of fertility behaviour which one may theoretically assume.

The study concludes with a new investigation which may highlight the interrelation between two useful variables viz. the period of post-partum amenorrhoea and the lactation period; which are useful items of consideration for averting birth without use of mechanical methods. This has been

done by a new application of renewal models which has been, hitherto, used in survival analysis, using Freund's Bivariate exponential model attempt has been made to construct the regression of Post Partum amenorrhoea period (P.P.A) on lactation period (when P.P.A. $>$ lactation period) for prediction of P.P.A; as well as regression of lactation on P.P.A. (when lactation period $>$ P.P.A).

The practical importance of Freund's model in this application is because of the assumption that with the ceasation of lactation the hazard rate for the discontinuance of lactation is also incr?ased given that the P.P.A has expired.

1.3 A Survey of the Outline of some of the Methodologies employed.

1. Palm Probability is defined as the conditional probability of a specified number of events in a time interval given that an event has happened at the beginning of the interval. Cox and Isham (1980) describe Palm Probability as follows :

For $u < v$, let $N(u,v)$ be a random variable giving the number of events occuring in (u,v) and X_i be the sequence of the intervals between i^{th} successive events $(i=1,2,3,\ldots\ldots)$ in a process starting from an arbitrary point.

Consider the survival function

$$K_X(x) = P(X > x), \text{ where X is a r.v. representing the waiting time for first renewal}$$

$$= \lim_{\delta \to 0+} P[N(0,x) = 0 \mid N(-\delta,o) > 0] \qquad (1,1)$$

(subject to orderliness which implies not more than one renewal in the infinitesimal interval) which is the limiting probability that, given an event occurs immediately before the origin, the next event of the process occurs after the instant (i.e. the system will survive more than a period x).

Now, by stationarity condition

$$\begin{aligned} & P[N(0,x) = 0 \cap N(-\delta,0) > 0] \\ &= P[N(0,x)=0] - P[N(-\delta, x)=0] \\ &= P[N(x)=0] - P[N(x+\delta)=0] \end{aligned} \qquad (1.2)$$

Now $P[N(0,x) = 0 \mid N(-\delta,0) > 0]$

$$= \frac{P[N(0,x)=0 \cap N(-\delta,0) > 0]}{P[N(-\delta,0)>0]}$$

$$= \frac{P[N(x)=0] - P[N(x+\delta)=0]}{P[N(\delta)>0]}$$

$$\therefore \quad P[N(0,x)=0 \mid N(-\delta,0)>0]\,\delta^{-1}P[N(\delta)>0]$$
$$= \delta^{-1}[P[N(x)=0]-P[N(x+\delta)=0]]$$
$$= -\delta^{-1}[P[N(x+\delta)=0]-P[N(x)=0]]. \qquad (1.2')$$

Define

$$\lim_{\delta \to 0+} \delta^{-1} P[N(\delta)>0] = \lambda \qquad (1.3)$$

as the occurrence parameter λ of the process which we assume as finite.

Further, denoting

$$P[N(x)=k] = P_k(x), \quad k=0,1,2,\ldots \qquad (1.4)$$

as the distribution of N (x), then in the limit as $\delta \to 0+$

$$K_X(x) = P(X>x)$$
$$= \lim_{\delta \to 0+} P[N(0,x)=0 \mid N(-\delta,0)>0]. \qquad (1.4')$$

Now $P[N(0,x)=0 \mid N(-\delta,0)>0]\,\delta^{-1}P[N(\delta)>0]$

$= K_X(x)\,\delta^{-1}P[N(\delta)>0]$ from (1.4′) as $\delta \to 0+$

Since the process is orderly, if $k > 0$, as $\delta \to 0+$

$$
\begin{aligned}
p_k(x+\delta) &= P[N(-\delta, x) = k] \\
&= P[N(-\delta, 0) = 0,\ N(0, x) = k] \\
&\quad + P[N(-\delta, 0) = 1,\ N(0, x) = k-1] + o(\delta) \\
&= p_k(x) - P[N(-\delta, 0) > 0,\ N(0, x) = k] \\
&\quad + P[N(-\delta, 0) > 0,\ N(0, x) = k-1] + o(\delta)
\end{aligned}
$$

so that

$$
\begin{aligned}
p_k(x+\delta) - p_k(x) &= -P[N(-\delta, 0) > 0,\ N(0, x) = k] \\
&\quad + P[N(-\delta, 0) > 0,\ N(0, x) = k-1] + o(\delta) \\
&= -P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k \mid N(-\delta, 0) > 0] \\
&\quad + P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k-1 \mid N(-\delta, 0) > 0] + o(\delta)
\end{aligned}
$$

$$
\begin{aligned}
\therefore\ \delta^{-1}[p_k(x+\delta) - p_k(x)] &= -\delta^{-1} P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k \mid N(-\delta, 0) > 0] \\
&\quad + \delta^{-1} P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k-1 \mid N(-\delta, 0) > 0] + o(1). \\
&= -\delta^{-1}\big[P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k \mid N(-\delta, 0) > 0] \\
&\quad - P[N(-\delta, 0) > 0] \cdot P[N(0, x) = k-1 \mid N(-\delta, 0) > 0] + o(1)
\end{aligned}
$$

$$\Rightarrow K_X(x)\lambda = D_X[P_0(x)] \quad \text{as } \delta \to 0 \tag{1.5}$$

where D_x denotes the derivative.

The equation (1.5) links the distribution of the interval between successive events with survivor function $K_x(x)$ to that of forward recurrence time with survivor function $P_0(x)$ (where $P_0(x)$ is the probability that starting from an arbitrary time instant, there are no events in the following interval of length x.

Palm Probability as a limiting probability :

(Palm's Integral Equation) : More general results connecting distributions of events conditional on a point at the origin with those where the origin is an arbitrary instant may be obtained. Assuming that the process is completely stationary and has a finite occurrence parameter ν and is orderly so that ν is equal to the rate λ of the process. Then for each x > 0, the palm distribution is a discrete distribution defined by

$$\pi_K(x) = \lim_{\delta \to 0+} P[N(0,x) = k \mid N(-\delta,0) > 0] \tag{1.6}$$

$$\text{for } k = 0, 1, 2, \ldots\ldots$$

In a careful mathematical development the existence of ν and $\pi_K(x)$, and more generally of other limiting

probabilities of events B given $N(-\delta,0) > 0$ of the form

$$\lim_{\delta \to 0+} P[B \mid N(-\delta, 0) > 0] \qquad (1.7)$$

has to be proved.

Let the probability measure Π be defined for events B on those processes which have a point at the origin. The measure Π can then be shown to satisfy

$$\Pi(B) = \lim_{\delta \to 0+} P[B \mid N(-\delta, 0) > 0] \qquad (1.8)$$

for a wide class of events B. The measure Π is called the palm measure of the process. In equation (1.8) the distribution of the interval measured from an arbitrary time instant to the next point of the process, is linked to that of the interval between successive points. In the same way the functions $\pi_k(x)$ defining the palm distributions given in (1.6), which specify the distribution of the number of events in an interval of length x given a point at the origin, can be connected with the functions $P_k(x)$ which give the distribution of the number of events in an interval of length x starts at an arbitrary origin. These connecting equations are known as Palm-Khintchine equations and may be derived as follows :

Hence

$$\delta^{-1}[p_k(x+\delta)-p_k(x)] = -\delta^{-1}[P[N(-\delta,0)>0]]\cdot[P(N(0,x)=k \mid N(-\delta,0)>0)]$$
$$- P[N(0,x)=k-1 \mid N(-\delta,0)>0] + o(1)$$

Hence taking limit as $\delta \to 0+$

$$D_k[p_k(x)] = \delta^{-1}P[N(-\delta,0)>0]\cdot[\pi_k(x)-\pi_{k-1}(x)]$$
$$= -[\pi_k(x) - \pi_{k-1}(x)] \text{ from (1.3) and (1.6)} \qquad (1.9)$$

where D_x denotes the right hand derivative. The corresponding equation for $k = 0$ has already been derived and is

$$D_x[p_0(x)] = -\lambda\pi_0(x). \qquad (1.10)$$

The integral forms of (1.9) and (1.10) are

$$p_k(x) = -\lambda\int_0^k[\pi_k(u) - \pi_{k-1}(u)]\,du \qquad (1.11)$$
$$k=1, 2, 3, \cdots\cdots$$

$$p_0(x) = 1 - \lambda\int_0^x \pi_0(u)\,du \qquad (1.12)$$

It follows from (1.9) and (1.10) that

$$-\frac{1}{\lambda}D_x[p_0(x)+\cdots\cdots+p_k(x)] = -\frac{1}{\lambda}D_x[P[N(x)\le k]]$$
$$= \pi_k(x)$$

Therefore, the probability of having exactly k events in (0,x) starting from an event at 0, can be obtained by differentiating the probability of getting not more than K events in (0,x) where 0 is an arbitrary time instant. Alternatively from (1.11) and (1.12),

$$p_0(x)+p_1(x)+\cdots\cdots+p_k(x) = P(N(x)\le k)$$
$$= 1-\lambda\int_0^x \pi_k(u)\,du \qquad (1.13)$$

so that the probability of getting not more than k events in (0,x], where 0 is an arbitrary instant can be obtained by integrating the probability of exactly k events in the interval when there is an event at 0. In addition the right hand side of (1.13) is equal to

$$1-\lambda\int_0^x \pi_k(u)\,du = \lambda\int_x^\infty \pi_k(u)\,du$$

$$\because\ p_0(x)+p_1(x)+\cdots\cdots+p_k(x) = 1-\lambda\int_0^x \pi_k(u)\,du$$

$$\Rightarrow p_0(\infty)+p_1(\infty)+\cdots\cdots+p_k(\infty) = 1-\lambda\int_0^\infty \pi_k(u)\,du$$

$$\Rightarrow \lambda\int_0^\infty \pi_k(u)\,du = 1$$

$$1-\lambda\int_0^x \pi_k(u)\,du = \lambda\int_0^\infty \pi_k(u)\,du - \lambda\int_0^x \pi_k(u)\,du$$

$$= \lambda\int_x^\infty \pi_k(u)\,du$$

Again

$$\lambda\int_x^\infty \pi_k(u)\,du = \lambda\int_0^\infty \pi_k(y+x)\,dy \quad \text{putting } u = y+x$$

Hence $p_0(x)+\cdots+p_k(x) = P[N(x)\le k] = \lambda\int_0^\infty \pi_k(y+x)dy$ (1.14)

(1.14) may be justified by the argument that if '0' is an arbitrary time instant and there are no more than k events in (0,x) then there must exist an event with coordinate y, for same $y > 0$, such that there are exactly k events in (-y,x). Since the process is orderly, the probability of an event in $(-y,-y+\delta)$ is $\lambda\delta+o(\delta)$

and therefore

$$P[N(x)\le k] = \lambda\int_0^\infty \pi_k(y+x)\,dy.$$

The equations (1.9), (1.10), (1.11) and (1.13) can be summarised by using probability generating functions. For if it is defined $G(Z,x) = \sum_{k=0}^\infty Z^k p_k(x)$

$$G_0(Z,x) = \sum_{k=0}^\infty Z^k \pi_k(x)$$

so that G refers to an arbitrary origin while G_0 refers

to the situation given a point at origin.

In fact

$$\sum_{i=k}^{\infty} \pi_i(x) = G^{(k)}(x)$$

where the right hand side denotes the cumulative distribution of $\left[X_1 + X_2 + \ldots\ldots + X_k\right]$ obtained by k-fold convolution.

That is

$$\pi_k(x) = G^{(k)}(x) - G^{(k+1)}(x)$$

and the $p_k(x)$ are given by

$$p_k(x) = -\lambda \int_0^x [\pi_k(u) - \pi_{k-1}(u)]\, du \qquad (1.15)$$

$$k = 1, 2, \cdots$$

$$p_0(x) = 1 - \lambda \int_0^x \pi_0(u)\, du \qquad (1.16)$$

<u>Birth and Death Process</u> is a process which permits a population to grow as well as to decline. These are more relevant to biological population in which both births and deaths occur. The birth and death process developed here is applied to some problems of policy making in sterilization in chapters 5 and 6 of the thesis.

Let X (t) = Size of the population at time t for $0 \leq t < \infty$

with the initial size $X(0) = k_o$

and $P_k(t) = 0$ where $X(t) = k$.

Given $X(t) = k$ we assume that the probability of exactly one birth in the interval $(t, t+\delta t)$ is $\lambda_o(t) + o(\delta)$

(i) The probability of exactly one death is

$$\mu_k(t) + o(\delta)$$

(ii) The probability of more than one change is $o(\delta)$.

Therefore, the probability of no change in $(t, t+\delta t)$ is

$$1 - \lambda_k(t) - \mu_k(t) + o(\delta).$$

Consequently, the probability $p_k(t+\delta)$ at some time $t+\delta$ may be given by the Kolmogorov equation

$$\begin{aligned} p_k(t+\delta) = {} & p_k(t)\,[1-(\lambda_k(t)+\mu_k(t)+o(\delta))] \\ & + p_{k-1}(t).\lambda_{k-1}(t).\delta + p_{k+1}(t).\mu_{k+1}(t).\delta \\ & + o(\delta) \end{aligned} \qquad (1.17)$$

This given us the corresponding system of differential equation

$$\frac{d}{dt}p_o(t) = -[\lambda_o(t)+\mu_o(t)].p_o(t) + \mu_1(t).p_1(t) \qquad (1.18)$$

$$\frac{d}{dt} p_k(t) = -[\lambda_k(t) + \mu_k(t)]\, p_k(t) + \lambda_{k-1}(t)\, p_{k-1}(t) + \mu_{k+1}(t) . p_{k+1}(t) \qquad k \geq 1 \tag{1.19}$$

Under initial condition

$p_{k_0}(0) = 1$ and $p_k(0) = 0$ for $k = k_0$.

(1.19) completely determines the probability distribution $P_k(t)$.

Appropriate assumption may be made regarding the function $\lambda_k(t)$ and $\mu_k(t)$ to obtain stochastic process corresponding to empirical phenomena.

A particular case : linear growth

$$\lambda_k(t) = k\lambda \ ; \quad \mu_k(t) = k\mu \tag{1.20}$$

$$\Rightarrow \frac{d}{dt} p_0(t) = \mu p_1(t)$$

$$\frac{d}{dt} p_k(t) = -k(\lambda + \mu)\, p_k(t) + (k-1).\lambda . p_{k-1}(t) + (k+1).\mu . p_{k+1}(t). \tag{1.21}$$

Employing the method of probability generating function (p.g.f) to solve the same

$$G_x(s,t) = \sum_{k=0}^{\infty} s^k p_k(t) \tag{1.22}$$

From (1.21) it follows

$$\frac{\partial}{\partial t} G_x(s,t) + (1-s)(\lambda s-\mu) \cdot \frac{\partial}{\partial s} G_x(s,t) = 0 \qquad (1.23)$$

With the initial condition $t = 0$ (1.24)

$$G_x(s,0) = s^{k_0}$$

The auxillary equations are (1.25)

$$dt = \frac{ds}{(1-s)(\lambda s-\mu)}$$

and $\quad dG_x(s,t) = 0$

For $\lambda \neq \mu$ we may use partial fractions to rewrite the first auxilliary equation as

$$dt = \frac{\lambda}{(\lambda-\mu)(\lambda s-\mu)} ds + \frac{1}{(\lambda-\mu)(1-s)} ds \qquad (1.26)$$

$$\Rightarrow (\lambda-\mu)\, dt = d \log\left[\frac{\lambda s-\mu}{1-s}\right] \qquad (1.27)$$

Integrating both sides of (1.27) =>

$$\frac{1-s}{s-\mu} e^{(\lambda-\mu)t} = \text{Constant}. \qquad (1.28)$$

The second auxilliary equation of (1.25) =>

$$G_x(s,t) = \text{Constant}. \qquad (1.29)$$

Therefore the general solution of (1.23) is

$$G_x(s,t) = \phi\left[\frac{1-s}{\lambda s-\mu}\right] \cdot e^{(\lambda-\mu)t} \qquad (1.30)$$

where ϕ is an arbitrary differentiable function.

Using the initial condition, it can be seen at t=0

$$\phi\left[\frac{1-s}{\lambda s-\mu}\right] = s^k \qquad (1.31)$$

at least for all s with $|s| < 1$.

Hence for all θ such that $|1+\theta\mu| < |1+\theta|$

$$\phi(0) = \left[\frac{1+\theta\mu}{1+\theta}\right]^{k_0}. \qquad (1.32)$$

Letting

$$\theta = \frac{1-s}{\lambda s-\mu} e^{(\lambda-\mu)t} \qquad (1.33)$$

The particular solution for the case of $\mu \neq \lambda$

given by

$$G_X(s,t) = \left[\frac{(\lambda s-\mu)+\mu(1-s)e^{(\lambda-\mu)t}}{(\lambda s-\mu)+\lambda(1-s)e^{(\lambda-\mu)t}}\right]^{k_0} \qquad (1.34)$$

for $\mu \neq \lambda$

Putting (1.35)

$$\alpha(t) = \mu \cdot \frac{1-e^{(\lambda-\mu)t}}{\mu-\lambda e^{(\lambda-\mu)t}}$$

$$\beta(t) = \frac{\lambda}{\mu} \cdot \alpha(t)$$

The p.g.f. can be rewritten as

$$G_X(s;t) = \left[\frac{\alpha(t) + [1-\alpha(t) - \beta(t)\,s]}{1 - \beta(t)s} \right]^{K_0} \quad \ldots (1\cdot 36)$$

Expanding the p.g.f. $G_X(s;t)$, $p_k(t) = P[X(t) = k]$ can be written as

$$p_k(t) = \sum_{j=0}^{\min[K_0,\,k]} \binom{K_0}{j} \binom{K_0 + k - j - 1}{k - j} (\alpha(t))^{K_0 - j} (\beta(t))^{k-j} \left[1 - \alpha(t) - \beta(t)\right]^{j}$$

$$\text{for } k \geq 1 \quad \ldots (1\cdot 37)$$

and

$$p_0(t) = [\alpha(t)]^{K_0} \quad \ldots (1\cdot 37')$$

The use of Martingales based on a density dependent birth and death process is done in chapter 6 on sterilisation policy.

A Stochastic Process X_n : $n = 0,1,2,3,\ldots\ldots$ is a martingale if for all $n = 0,1,2,3,$

(i) $E\{|X_n|\} < \infty$

(ii) $E\{X_{n+1} \mid X_0, X_1 \ldots X_n\} = X_n \quad \ldots (1\cdot 38)$

hold

A more general definition :

Let X_n : $n = 0,1,2,3\ldots\ldots$.

and $(Y_n : n = 0,1,2,3,......)$ are stochastic processes.

We say $\{X_n\}$ is a Martingale with respect to $\{Y_n\}$ if for all $n = 0,1,2,3......$

(i) $E(|X_n|) < \infty$

(ii) $E\{X_{n+1} \mid Y_0, Y_1, \cdots, Y_n\} = Y_n$ (1.39)

One may imagine $(Y_0, Y_1 ...., Y_n)$ as the information history upto the stage n.

Wald's Martingale :

Let $\phi(\lambda) = E[e^{\lambda Y_k}]$ exists for some $\lambda \neq 0$ where (Y_k)'s are independent identically distributed random variables and $Y_0 = 0$ for $K = 0,1,2,3,......$ then

$$X_n = (\phi(\lambda))^{-n} e^{\lambda(Y_1 + Y_2 + Y_3 + \cdots + Y_n)}$$

is a Martingale with respect to Y_n

Sub Martingale and Super Martingale :

Let $\{X_n : n = 0, 1, 2, \cdots\}$ and $\{Y_n : n = 0, 1, 2, \cdots\}$ be stochastic processes then (X_n) is called a Super

Martingale with respect to (Y_n) if for all n

(i) $E(X_n^-) > -\infty$ where $X^- = \inf\{X, 0\}$

(ii) $E[X_{n+1} | Y_0, Y_1, \ldots, Y_n] \leq X_n$

(iii) X_n is a function of $Y_0, Y_1, Y_2, \ldots, Y_n$.

Similarly (X_n) is called a Sub Martingale with respect to (Y_n) if :-

(i) $E[X_n^+] < \infty$ Where $X^+ = \mathrm{Sup}(X, 0)$

(ii) $E[X_{n+1} | Y_0, Y_1, Y_2, \ldots, Y_n] \geq X_n$

(iii) X_n is a function of $Y_0, Y_1, \ldots, Y_n$.

Definition of Markov time (or Stopping time) :

A random variable T associated with a Martingale (X_T) is called a Markov time with respect to (Y_n) if T takes the value 0,1,2,...... and if for all n = 0,1,2,.... the event T = n is determined by $(Y_0, Y_1, \ldots, Y_n)$.

By the term 'determined' implied that the indicator function of the event (T = n) can be written as a function of $Y_0, Y_1, Y_2, \ldots, Y_n$.

Optional Sampling Theorem :

Suppose (X_n) is a Martingale and T is a Markov time with respect to (Y_n) then

$$E(X_0) = E(X_{T\cap n}) = \lim_{n\to\infty} E(X_{T\cap n}). \qquad (1.40)$$

If $T < \infty$ then $\lim_{n\to\infty} X_{T\cap n} = X_T$.

Actually $X_{T\cap n} = X_T$ whenever $n > T$.

Thus whenever we can justify the interchange of limit $n \to \infty$ and expectations exists the following result holds

$$E[X_0] = \lim_{n\to\infty} E[X_{T\cap n}] = E[\lim_{n\to\infty} X_{T\cap n}] = E(X_T) \qquad (1.41)$$

Optional Stopping Theorem

Let (X_n) be a Martingale and T be a Markov time.

If

(i) $P[T<\infty] = 1$

(ii) $E[|X_T|] < \infty$

(iii) $\lim_{n\to\infty} E[X_n I_{(T>n)}] = 0$

$$E(X_T) = E(X_0) \qquad (1.42)$$

where $I_{(T>n)}$ represents the indicator function for $T > n$.

In Chapter 7 Counter Theory is employed to evaluate the direct and indirect strategies for the reduction of conception rate.

A Geiger-Muller Counter Model I is a registering mechanism for detecting the presence of a radio active

material arriving at the counter, but because of inertia, the counter will not register some of the impulses. More specifically, suppose that an impulse arrives at a fixed time t the counter registers the impulse. The registration causes a dead time say of length π_1 and impulses during the dead time will not be registered by the counter. In general, the first impulse to arrive after the termination of the dead time π_1 again will be registered by the counter andthis again causes a dead time of length say π_2 and so on. However, in the simpler cases, the impulses which arrive during a dead time do not cause any dead time so that each dead time is caused by a registered impulse. This is called a Geiger-Muller Counter of type I. (Pyke 1958). In a counter of type II (Smith 1958) each arriving impulse causes a dead time so that arrivals during a period of dead time prolong further the dead time. A counter model although basically a description of certain physical processes, can be applied to many Biological, Social and Industrial processes.

Counter model of type I with fixed dead time x

We illustrate the counter model type I with fixed dead time using the following problem :

A conception takes place with intensity λ subject to the condition that every conception is followed by infecundable exposure x (fixed dead time) during which no further conception takes place. Then to obtain the probability distribution of the number of conceptions in $(o,t]$.

Let $\tau_1 < \tau_2 \cdots < \tau_n \cdots$ be the renewal times (waiting time of conceptions) with conception rate λ (Poisson intensity) having negative exponential density function

$$f(t) = \lambda e^{-\lambda t} \quad ; \quad 0 \leq t < \infty \quad ; \lambda > 0$$

$$\Rightarrow \quad P[\tau_1 \leq x] = 1 - e^{-\lambda x} \; ; \; \lambda \geq 0 \; , \; 0 \leq x < \infty$$

$$P[\tau_n - \tau_{n-1} \leq x] = 1 - e^{-\lambda(x - \pi)} \quad \text{for } x \geq \pi$$
$$= 0 \qquad \text{otherwise} \qquad (1.43)$$

Let W (t,n) = Probability of not more than n events (conceptions) up to time t

$$w(t,n) = P[\tau_{n+1} > t]$$
$$= 1 - P[\tau_{n+1} \leq t]$$
$$= 1 - F_{n+1}(t)$$
$$= R_{n+1}(t) \qquad (1.44)$$

where $F_{n+1}(.)$ is the c.d.f. of the random variable τ_{n+1} and $R_{n+1}(t) = 1 - F_{n+1}(t)$ is the corresponding survival function.

Denoting L (.) as the Laplace transform

$$L(w(t,n)) = \int_0^\infty e^{-st} (1 - F_{n+1}(t))\, dt$$
$$= \frac{1}{s} - \frac{1}{s} L(F_{n+1}(t)) \qquad (1.45)$$

($\because$ $L(F_{n+1}(t)) = \frac{1}{S} L(f_{n-1}(t))$ where

$f_{n+1}(.)$ is the interval density function of $(n+1)^{th}$ renewal (order of conceptions).

We have $f_{n+1}(t) = f_1 * \{f(n)\}^*$

where f_1 is the density function of τ_1, f is the density function of $(\tau_r - \tau_{r-1})$, $r \geq 2$,

and $*$ stands for convolution;

and $f(n)^* \equiv$ n fold convolution of f

$$\Rightarrow L(f_{n+1}(t)) = L(f_1(t))(L(f(t))^n)$$

Now $L(f_1(t)) = \frac{\lambda}{\lambda + S}$

and $L(f(t)) = \int_{\pi}^{\infty} e^{-st} \lambda e^{-\lambda(t-\pi)} dt$

$$= \frac{\lambda e^{-\pi s}}{s+\lambda} \qquad (1.46)$$

$$\therefore \quad L(f_{n+1}(t)) = \frac{\lambda^{n+1}}{(s+\lambda)^{n+1}} e^{-n\pi s} \qquad (1.47)$$

Putting (1.46) and (1.47) in (1.45) $\Rightarrow$

$$L(W(t,n)) = \frac{1}{s} - \frac{1}{s}\,\frac{\lambda^{n+1}}{(s+\lambda)^{n+1}}\,e^{-n\pi s} \qquad (1.48)$$

By taking inverse Laplace Transform

$$W(t,n) = 1 - \lambda \int_{n\pi}^{t} \frac{e^{-\lambda(u-n\pi)}\,\lambda^{n}\,(u-n\pi)^{n}}{\Gamma(n+1)}\,du \qquad (1.49)$$

Also using the result

$$\sum_{j=0}^{n} e^{-M}\,\frac{M^{j}}{j!} = 1 - \int_{0}^{M} \frac{e^{-z}z^{n}}{\Gamma(n+1)}\,dz \qquad (1.50)$$

Putting $M = \lambda(t-n\pi)$ on both sides of (1.50)

$$\sum_{j=0}^{n} e^{-\lambda(t-n\pi)}\,\frac{(\lambda(t-n\pi))^{j}}{j!} = 1 - \int_{0}^{\lambda(t-n\pi)} \frac{e^{-z}z^{n}\,dz}{\Gamma(n+1)}$$

Further on substitution of $z = \lambda(u-n\pi)$

$$z = 0 \Rightarrow u = n\pi$$

$$z = \lambda(t-n\pi) \Rightarrow u = t$$

$$\Rightarrow \sum_{j=0}^{n} e^{-\lambda(t-n\pi)}\,\frac{(\lambda(t-n\pi))^{j}}{j!}$$

$$= 1 - \frac{\lambda}{\Gamma(n+1)} \int_{n\pi}^{t} e^{-\lambda(u-n\pi)}\,(\lambda(u-n\pi))^{n}\,du \qquad (1.51)$$

Comparing (1.51) with (1.49) $\Rightarrow$

$$W(t;n) = \sum_{j=0}^{n} e^{-\lambda(t-n\pi)} \frac{(\lambda(t-n\pi))^j}{j!} \qquad (1.52)$$

$$P[X=n] = W(t;n) - W(t;n-1)$$

$$= \sum_{j=0}^{n} e^{-\lambda(t-n\pi)} \frac{(\lambda(t-n\pi))^j}{j!}$$

$$- \sum_{j=0}^{n-1} e^{-\lambda(1-(n-1)\pi)} \frac{(\lambda(t-(n-1)\pi))^j}{j!} \qquad (1.53)$$

and $n \leq \left[\frac{t}{\pi}\right]$ where $\left[\frac{t}{\pi}\right]$ refers to the greatest integer contained in $\frac{t}{\pi}$

<u>Counter Model Type I with Random Variable Dead Time</u>

We illustrate the concept of counter model type I with the following problem :

Problem : A particle arrives at t=0 and locks the counter for a dead time of duration Y_1. With the registration of the particle the counter is blocked for a time of length say Y_2. The next particle to be registered is that of the first arrival once the counter is fixed. The process is repeated where the successive locking times denoted by Y_1, Y_2, Y_3 are assumed to be independent with a common distribution

$$P[Y_k < y] = G(y)$$

and independent of the arrival process. We obtain the waiting time distribution of the inter-arrival of the process.

Let $Z = Y + Y_y$.

Where Y is the dead time following the registration and Y_y is the residual life time before the counter is locked.

Where Y is the dead time and Y_y be the residual life time before the counter is locked.

Denoting by $g(.)$ and $f(.)$ the density functions of the dead time and the residual free time (before the system is again being locked) have

$$P\{z \leq Y + Y_y \leq z + dz\}$$

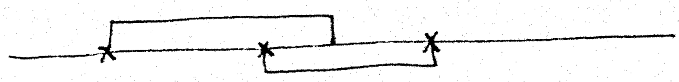
$$= \psi(z)\,dz$$

$$= \int_0^z g(y)\, f(z-y)\, dy$$

Counter Model Type II with Fixed Dead Time

Here the locking mechanism is more complicated. As before, an incoming signal is registered if and only if it arrives when the counter is free. In type I counter only recorded particles induced the counter to lock. For Type II Counter every arriving signal can prolong the dead time in the counter, the associated locking times

being added concurrently



$*$ ≡ Arrival (registration)

$\square$ ≡ Dead time

Figure :

Let $\tau_1 < \tau_2 < \cdots < \tau_n$ are the registration times of the first, second and the nth renewal and arrivals take place with Poisson rate λ. The length of the dead time following every registration is π.

We assume $\{\tau_n - \tau_{n-1}\}$ are the independent identically distributed random variables.

$$P[\tau_1 \leq x] = 1 - e^{-\lambda x} \tag{1.55}$$

$$P[\tau_n - \tau_{n-1} \leq x] = F(x) \tag{1.56}$$

where the structure of F(x) is to be obtained.

$$\text{Let } P[x \leq \tau_n - \tau_{n-1} \leq x + dx] = f(x)\,dx \tag{1.57}$$

$$\phi(s) = \int_0^\infty e^{-st} f(t)\,dt = L[f(t)]$$

Let $u(t) = \frac{dU(t)}{dt}$ be the renewal density of the process and U(t) be the renewal function.

Then the Laplace transform of the Renewal Density is given by

$$L(u(t)) = \frac{L(f_1(t))}{1 - L(f(t))}$$

$$L(f_1(t)) = \lambda \int_0^\infty e^{-st} e^{-\lambda t} dt = \frac{\lambda}{\lambda + s} \qquad (1.58)$$

Also $$L(u(t)) = L\left(\frac{dU(t)}{dt}\right) = \frac{\frac{\lambda}{\lambda+s}}{1 - \phi(s)}$$

$$\Rightarrow 1 - \phi(s) = \frac{\lambda}{\lambda+s}\left[\int_0^\infty \left[e^{-st}\frac{dU(t)}{d(t)} d(t)\right]\right]^{-1}$$

$$\Rightarrow 1 - \frac{\lambda}{\lambda+s}\left[\int_0^\infty e^{-st}\frac{dU(t)}{d(t)} d(t)\right]^{-1} = \phi(s) \qquad (1.59)$$

To obtain $\frac{dU(t)}{dt}$, we proceed as follows :

$$U(t+dt) - U(t) = e^{-\lambda t}\lambda\delta t + (\delta t) \quad \text{if } t < \pi \qquad (1.60)$$

given that initially the counter is free.

Also $$U(t+dt) - U(t) = e^{-\lambda\pi}\lambda\delta t + o(\delta t) \quad \text{if } t \geq \pi \qquad (1.61)$$

$$\Rightarrow \frac{dU(t)}{dt} = \lambda e^{-\lambda t} \quad \text{if } t < \pi \qquad (1.62)$$

$$= \lambda e^{-\lambda\pi} \quad \text{if } t \geq \pi$$

$$\therefore L\left(\frac{du(t)}{dt}\right) = \int_0^\infty e^{-st}\,\frac{du(t)}{dt}\,dt$$

$$= \int_0^\pi e^{-st}\,\frac{du(t)}{dt}\,dt + \int_\pi^\infty e^{-st}\,\frac{du(t)}{dt}\,dt$$

$$= \lambda\int_0^\pi e^{-st}\,e^{-\lambda t}dt + \int_\pi^\infty e^{-st}\,e^{-\lambda\pi}\,dt$$

$$= \lambda\int_0^\pi e^{-t(\lambda+s)}\,dt + \lambda e^{-\lambda\pi}\int_\pi^\infty e^{-st}\,dt$$

$$L(U(t)) = \frac{\lambda e^{-\lambda(\lambda+s)\pi}}{s} - \frac{\lambda e^{-(\lambda+s)\pi}}{\lambda+s} + \frac{\lambda}{\lambda+s} \qquad (1.63)$$

$$\Rightarrow \quad u\ (t) = \frac{dU\ (t)}{dt}$$

$$= L^{-1}\left(\frac{\lambda}{\lambda+s} + \lambda e^{-\lambda\pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right)e^{-s\pi}\right) \qquad (1.64)$$

$$L^{-1}\left(\frac{\lambda}{\lambda+s}\right) = \lambda e^{-\lambda t}$$

$$L^{-1}\left(\lambda e^{-\lambda\pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right)\right) = \lambda e^{-\lambda\pi}(1 - e^{-\lambda t})$$

$$L^{-1}\left(\lambda e^{-\lambda\pi}\left(\frac{1}{s} - \frac{1}{\lambda+s}\right)e^{-s\pi}\right) = \lambda e^{-\lambda\pi}(1 - e^{-\lambda(t-\pi)}) \quad \text{if } t > \pi$$

$$= \lambda e^{-\lambda\pi} \text{ if } t \leq \pi$$

by the second shifting property of Laplace Transform.

$$\therefore\ U(t) = \lambda e^{-\lambda t} + \lambda e^{-\lambda\pi}(1 - e^{-\lambda(t-\pi)}) \quad \text{if } t \geq \pi$$

$$= \lambda e^{-\lambda t} + \lambda e^{-\lambda\pi} \qquad \text{if } t < \pi \qquad (1.65)$$

$$L(U(t)) = \frac{1}{s} L\left(\frac{dU(t)}{dt}\right) = \frac{1}{s} L(U(t))$$

$$= \frac{1}{s}\left[\frac{\lambda}{\lambda+s} + \lambda\left(\frac{1}{s} - \frac{1}{\lambda+s}\right) e^{-(\lambda+s)\pi}\right]$$

$$\Rightarrow U(t) = (1 - e^{-\lambda t}) + \lambda e^{-\lambda\pi}\left((t-\pi) - \frac{1}{\lambda}(1 - e^{-\lambda(t-\pi)})\right)$$

$$\text{if } t \geq \pi$$

$$= 1 - e^{-\lambda t} \qquad \text{if } t < \pi \qquad (1.66)$$

<u>Counter Model Type II with Random Variable Dead Time</u>

This counter process is quite difficult to analyse in general form. However, the results are known only for process with Poisson inputs having arrival rate λ .

Let $p(t)$ be the probability that the counter is free at a time t/i and registration is possible.

It is to show that

$$P(t) = e^{-\lambda} \int_0^t [1 - G(y)]\,dy \qquad (1.67)$$

where G(y) represents the cumulative density function of the dead time distribution.

Given n occurences of a Poisson process in the interval (0,t] the distribution of occurence times is the same as that of n independent random variables taken from a uniform distribution in (0,t].

Proof : The counter is free at time T = t if and only if all dead periods engendered by these signals have been terminanted before t.

Let G(t-y) = P (dead time commencing in y will end before the time t).

$$P\left[\begin{matrix}\text{Induced period}\\ \text{culminated prior to } t\end{matrix}\right] = \int_0^t G(t-y)\,dy \Big/ \int_0^t dy \qquad (1.68)$$

Since the locking times are assumed to be independent and also independent of the arrival process.

We have,

$$P\left[\begin{matrix}\text{counter is free}\\ \text{at time } t\end{matrix} \;\middle|\; \begin{matrix}n \text{ signals}\\ \text{in } (0,t]\end{matrix}\right] = \left[\frac{\int_0^t G(t-y)\,dy}{\int_0^t dy}\right]^n \qquad (1.69)$$

But the number of signals arriving during the interval $(0,t]$ has a Poisson distribution with mean λt.

From the low of total probability

$$p(t) = \sum_{j=0}^{\infty} \left\{ \frac{1}{t} \int_0^t G(t-y)\,dy \right\}^j \frac{(\lambda t)^j e^{-\lambda t}}{j!}$$

$$= \sum_{j=0}^{\infty} \left\{ \frac{\lambda t}{t} \int_0^t G(t-y)\,dy \right\}^j \frac{e^{-\lambda t}}{j!} \qquad (1.70)$$

$$= e^{-\lambda t} \left\{ \sum_{j=0}^{\infty} \lambda \int_0^t G(t-y)\,dy \right\}^j / j!$$

$$= e^{-\lambda t} \, e^{\lambda \int_0^t G(t-y)\,dy}$$

$$= e^{-\lambda \int_0^t dy} \, e^{\lambda \int_0^t G(t-y)\,dy}$$

$$= e^{-\lambda \int_0^t dy} \, e^{\lambda \int_0^t G(\tau)\,d\tau}$$

$$\therefore \quad p(t) = e^{-\lambda \int_0^t (1-G(y))\,dy} \qquad (1.71)$$

Counting with our assumption $\lambda\, p(t)$ is the probability density.

$$\Rightarrow \quad \frac{dU(t)}{dt} = \lambda\, p(t)$$

where U(t) represent the renewal function in $(0,t]$

P (a signal appearing in $(t, t+\delta t)$)

$$= p(t)\,\lambda\delta t + o(\delta t)$$

$$\therefore\; U(t+\delta t) = U(t) + [1.\lambda(\delta t)\, p(t) + 0(1-p\lambda\delta t)] + o(\delta t)$$

$$\lim_{t\to 0} \frac{U(t+\delta t) - U(t)}{t} = \lambda\, p(t)$$

$$\text{i.e.} \quad \frac{dU(t)}{dt} = \lambda\, p(t)$$

$$U(t) = \lambda \int_0^t p(\tau)\, d\tau \qquad (\because U(0) = 0). \qquad (1.72)$$

Putting (1.71) in (1.72)

$$U(t) = \lambda \int_0^t e^{-\lambda \int_0^{\tau} (1-G(y))dy}\, d\tau \qquad (1.73)$$

This proves the result.

CHAPTER - 2

ON THE APPLICATION OF PALM PROBABILITY FOR OBTAINING INTER-ARRIVAL TIME DISTRIBUTION IN WEIGHTED POISSON PROCESS

2.1 The distribution of $z_n = \{T_n - T_{n-1}\}$; $n = 1, 2, 3, \ldots$ where T_n represents the time for the n^{th} renewal in a Poisson process with parameter λ is given by the density function.

$$f(t|\lambda) = \lambda e^{-\lambda t} \quad ; \; 0 \leq t < \infty \; ; \; \lambda > 0 \tag{2.1}$$

However, if λ, instead of being a constant assumes a probability distribution with density

$$\psi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1} \tag{2.2}$$

then the unconditional distribution of time till the first renewal (or arrival) is given by

$$\int_0^\infty f(t|\lambda)\, \psi(\lambda)\, d\lambda = \frac{k a^k}{(a+t)^{k+1}} \tag{2.3}$$

But the waiting time distribution for the n^{th} arrival given that the first took place at $T=t$ is not certainly the n-fold convolution of (2.3). Because of weighting, the renewal

structure of the Poisson process is completely destroyed leading to interarrival distributions having infinitely divisible structure but with dependent increments. Denoting by $X(t)$ the number of events in $(0,t]$ from a Poisson process weighted by a Gamma distribution given by (2.2) it follows that $\forall\ t > s$

$$\begin{aligned} \mathrm{Cov}\,(X(s),\, X(t)-X(s)) \\ &= E\left[X(s)\, X(t)-X(s) \mid \lambda\right] - [E(\lambda)]^2 s(t-s) \\ &= E(\lambda^2 s(t-s)) - [E(\lambda)]^2 s(t-s) \\ &= s(t-s)\,\mathrm{Var}(\lambda) > 0 \qquad (2.4) \end{aligned}$$

This makes the problem of obtaining the probability distribution of the interarrival time in a compound Poisson process (or any such compound dependent process) somewhat complicated. On the other hand such interarrival time distributions are often considered as very useful for practical purposes; say while obtaining the distribution of interpregnancy (or interbirth) intervals or the waiting time distribution between two morbidity spells or the interarrival time distribution between two accidents. In all the cases the hazard rate $\lambda(t)$ (or the intensity) even for the same t varies between individuals.

Cox and Isham (1980) have dealt with a series of such problems on dependent process classifying the problems according to the nature of dependence. This chapter is devoted to exhibiting an application of Palm probability to obtain the interarrival distribution between the first and the r^{th} arrival ($r = 2,3,\ldots\ldots$) for a dependent process.

2.2 The Result :

Defining Palm probability $\psi_r(t)$ as the conditional probability of r events ($r = 0,1,2,3\ldots\ldots$) in $(0,t]$ given that an event has occurred at time $T=0$, Khinchin (1960) gives

$$V_r(t) = \frac{k}{a}\int_0^t [\psi_{r-1}(\tau) - \psi_r(\tau)]\, d\tau \qquad (2.5)$$

and

$$V_0(t) = 1 - \frac{k}{a}\int_0^t \psi_0(\tau)\, d\tau \qquad (2.6)$$

where $V_r(t)$ represents the unconditional probability of r events (arrivals) in $(0,t]$ following a weighted Poisson process (weighted by Gamma distribution as in (2.2) and

$$\frac{k}{a} = \lim_{t\to 0} \frac{1 - V_0(t)}{t} \qquad (2.7)$$

is the intensity of the process (Khinchine, (1960)).

Then

$$V_0(t) = E_\lambda(e^{-\lambda t}/\lambda) = \int_0^\infty e^{-\lambda t}\psi(\lambda)d\lambda = \frac{a^k}{(a+t)^k} \quad \cdots (2.8)$$

Using (2.6) and (2.7) we have

$$\left(\frac{a}{a+t}\right)^k = 1 - \frac{k}{a}\int_0^t \psi_0(\tau)\,d\tau$$

which on differentiation gives

$$\frac{a^{k+1}}{(a+t)^{k+1}} = \psi_0(t) \qquad (2.9)$$

Again $\psi_0(t) = P[T_1 > t \mid$ an event has occurred at $T = 0]$. Where T_1 is the random time of the first event following the occurrence of an event at T=0. Hence

$$f_1(t|\cdot) = \frac{d}{dt}(1 - \psi_0(t)) = \frac{(k+1)\,a^{k+1}}{(a+t)^{k+2}} \qquad (2.10)$$

where $f_1(t/\cdot)$ is the conditional density of the waiting time of the second event given that the first has occurred at T = 0. (2.10) is obviously different from the unconditonal waiting time density of the first arrival given in (2.3). Proceeding in this way, putting r=1 and differentiating

both sides of (2.5) we have

$$\psi_1'(t) = \frac{k}{a}\left[\psi_0(t) - \psi_1(t)\right]$$

$$\Rightarrow \frac{d}{dt}\left[k\left(\frac{a}{a+t}\right)^k\left(\frac{t}{a+t}\right)\right] = \frac{k}{a}\left[\frac{a^{k+1}}{(a+t)^{k+1}} - \psi_1(t)\right]$$

$$\Longrightarrow \psi_1(t) = \frac{(k+1)\,a^{k+1}\,t}{(a+t)^{k+2}} \qquad \cdots (2.11)$$

Then,

$$F_2(t|\cdot) = 1 - \psi_0(t) - \psi_1(t). \qquad (2.12)$$

where $F_2(t|\cdot)$ is the conditional cumulative distribution function (c.d.f) of the waiting time distribution of the second arrival given that at T=0 the first arrival took place.

Then,

$$f_2(t|\cdot) = \frac{a^{k+1}(k+1)}{(a+t)^{k+2}} - \frac{a^{k+2}(k+2)}{(a+t)^{k+3}} - \frac{k\,a^{k+1}}{(a+t)^{k+2}} + \frac{k(k+2)\,t\,a^{k+1}}{(a+t)^{k+2}} + \frac{a^{k+1}(k+1)}{(a+t)^{k+2}}$$

$$= \frac{t\,a^{k+1}(k+1)(k+2)}{(a+t)^{k+3}} \qquad \cdots (2.13)$$

which is the interarrival density function between the first and the third arrival given that the first arrival took place at T=0 in a compound Poisson process. Proceeding precisely in the same way we have,

$$V_2'(t) = \frac{k}{a}\left[\psi_1(t) - \psi_2(t)\right] \quad \cdots (2.14)$$

$$\Rightarrow \frac{d}{dt}\left\{\frac{(k+1)^k}{2}\left(\frac{a}{a+t}\right)^k\left(\frac{t}{a+t}\right)^2\right\} = \frac{k}{a}\left[\frac{(k+1)a^{k+1}}{(a+t)^{k+1}} - \psi_2(t)\right]$$

$$\Rightarrow \psi_2(t) = \frac{t^2 \, a^{k+1} \, (k+1)(k+2)}{2! \, (a+t)^{k+3}} \quad \cdots (2.15)$$

$$\Rightarrow F_3(t|\cdot) = 1 - \left[\psi_2(t) + \psi_1(t) + \psi_0(t)\right]$$

$$= 1 - \left[\frac{a^{k+1}(k+1)(k+2)t^2}{2!\,(a+t)^{k+3}} + \frac{a^{k+1}\,t(t+k)}{(a+t)^{k+2}} + \frac{a^{k+1}}{(a+t)^{k+1}}\right]$$

when we get

$$f_3(t|\cdot) = \frac{a^{k+1}(k+2)(k+3)t^2}{2!\,(a+t)^{k+4}} \quad \cdots (2.16)$$

Using

$$f_n(t|\cdot) = \frac{d}{dt}\left\{1 - \sum_{r=0}^{n-1}\psi_r(t)\right\} \quad \cdots (2.17)$$

and the recurrence relation

$$f_n(t|\cdot) = f_{n-1}(t|\cdot) - \psi_{n-2}(t) + \frac{a}{k}V_{n-1}(t) \quad \cdots (2.18)$$

we have

$$f_n(t|\cdot) = \frac{t^{n-1}\ \alpha^{k+1}(k+1)(k+2)\cdots(k+n)}{(n-1)!\ (\alpha+t)^{k+n+1}} \quad \cdots (2.19)$$

which provides the distribution of the time between the first and the $(n+1)^{th}$ arrival given that the first arrival occurred at T=0 for a compound Poisson process weighted by a Gamma distribution.

The treatment for other dependent processes is precisely the same as illustrated in the foregoing example.

<u>Remarks</u> :

If $(T_i - T_{i-1})$ is the waiting time for the i^{th} order of conception $(i = 2,3,\ldots..)$ or $(T_i - T_{i-1})$ may be called 'Interconception' interval and the conception rate λ (even taken independent of i) varies from individual to individual conforming to some probability distribution say Gamma distribution, then because of waiting λ

(i) Renewal intervals $(T_i - T_{i-1})\ \forall\ i = 2,3,\ldots\ldots$ will cease to become $i.i.d.r.v.^s$ (or i.d.r.v.s.) contrary to the traditional assumption.

(ii) The number of renewals even in two non overlapping intervals will be correlated. The extent of correlation because of the weighting of the process. The renewal structure is completely destroyed; leading to the process to conform to infinitely divisible distribution with dependent increments.

Correlation between T_i and $T_{j-i} = T_j - T_i$ $\quad j > i$.

Let us assume the probability distribution of λ to be

$$\phi(\lambda) = \frac{a^K}{\Gamma(K)} e^{-a\lambda} \lambda^{k-1} \quad ; \quad 0 \leq \lambda < \infty, \quad a, K > 0$$

We have

$$\mathrm{Cov}(T_i, T_j) = E[E(T_i T_j \mid \lambda)] - E(E(T_i \mid \lambda))(E(E(T_j \mid \lambda)))$$

$$E(T_i T_j \mid \lambda) = E(T_i (T_j + T_{j-i} \mid \lambda))$$

$$= E(T_i^2 \mid \lambda) + [\mathrm{Cov}(T_i, T_{j-i} \mid \lambda) + E(T_i \mid \lambda) E(T_{j-i} \mid \lambda)]$$

Now $\mathrm{Cov}(T_i, T_{j-i}) = 0$.

Since T_i and T_{j-i} being two non overlapping intervals.

$$E(T_i^2 \mid \lambda) = \frac{i(i+1)}{2}, \quad E(T_i \mid \lambda) = \frac{1}{\lambda}$$

and $$E(T_{j-i} \mid \lambda) = \frac{j-i}{\lambda}$$

$$\therefore \quad E(T_i T_j \mid \lambda) = i(j+1)\, E\left(\frac{1}{\lambda^2}\right).$$

Since $$E\left(\frac{1}{\lambda^2}\right) = \frac{a^2}{(K-1)(K-2)} \quad ; \quad K > 2$$

$$E(T_i T_j) = \frac{i(j+1)\, a^2}{(k-1)(K-2)}$$

$$\mathrm{Cov}(T_i T_j) = \frac{a^2 i(j+k-1)}{(K-1)^2 (K-2)} \quad ; \quad k > 2$$

$$\mathrm{Var}(T_i) = \frac{a^2 i(i+k-1)}{(k-1)^2 (k-2)} \quad ; \quad k > 2$$

$$\mathrm{Cov}(T_i, T_j) = \mathrm{Cov}(T_2, T_i + T_{j-i}).$$

CHAPTER-3

On the Comparison of Cohort Fertilities by Palm Probabilistic Technique

3.0 In this chapter a suitable methodology is evolved to predict and then to compare the future fertility performance (vis-a-vis the total fertility status). Given the time of occurrence of first birth as t_1 and t_2 respectively of the two cohorts, whose age of marriage differs by a time lag say, γ it is proposed to analyse the level of fertility changes between two cohorts by the average number of births to these cohorts of women given that the first birth to the first cohort of women took place at t_1 and the first birth of the second cohort of women took place at $t_2 (t_2 > t_1)$ leaving the residual fertility span $(T-t_1)$ & $(T-t_2)$ respectively for the two classes of women who complete their fertility span at the age T.

The purpose is to see whether postponement of the age at marriage is effective in reducing the fertility status by using the technique of Palm Probability.

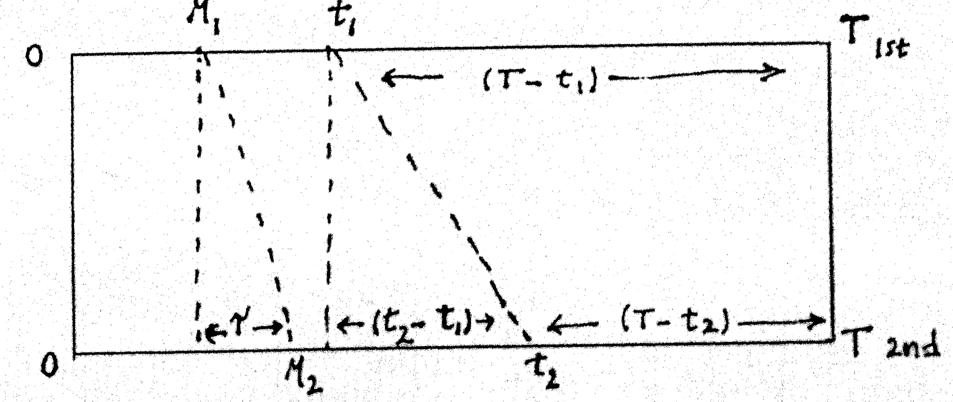


Figure 3.1

Palm Probability

Stochastic modelling involving prediction of the number of events during a fixed period of time (0,t) or the prediction of the waiting time distribution between two consecutive events or between any two events say the ith and the jth event can be solved by an entirely new technique known as "Palm Probability". It is the conditional probability of a specific number of events given that the event has occurred at the beginning of the interval.

Let us consider two consecutive intervals of length s and (t-s) respectively as shown in the following figure :

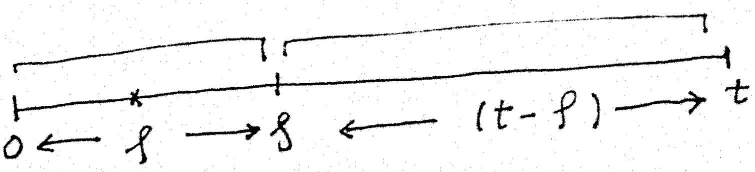


Figure 3.2

Further let the interval (0,s) be so small such that at most one arrival of the shock with compound Poisson inputs (i.e. Poisson input λ weighted by a two parameter family of Gamma distribution for taking the variation of fecundability between individuals) is possible and let us assume that there is only one arrival in (0,s) without any loss of generality. Then as $s \to 0$, the point of arrival denoted by X (in Figure 3.2) in the limiting position will be located in the beginning of the interval (s,t). What is the probability of having

a finite number of arrivals of shocks, say k number of arrivals (k=0,1,2,.....) in the interval (s,t) of any finite length given an arrival at the beginning of s? What is the waiting time distribution of next arrival or the nth arrival in sequence following that event (i.e. an event has occurred at s). These limiting conditional probabilities are known as Palm Probabilities.

Let $\phi_k(t)$ be the conditional probability of k number of births in (0,t) given that a birth has occurred at t=0 ($\phi_k(t)$ is a palm probability measure)

and $V_k(t)$ be the unconditional probability of k births in (0,t); k = 0,1,2,........ .

Then Palm's integral equations connecting $V_k(t)$ and $\phi_k(t)$ are as follows (Khintchine (1960)

$$V_k(t) = \int_0^t [\phi_{k-1}(\tau) - \phi_k(\tau)]\, d\tau \qquad k = 0,1,2,\ldots \qquad \cdots (3.1)$$

and

$$V_0(t) = h(t) = h(t) \int_0^t \phi_0(\tau)\, d\tau \qquad \cdots (3.2)$$

where h (t) represents the intensity of the process that the hazard rate of having a birth at the age t.

Since the hazard rate h (t) is a decreasing function

of t for a given woman with fecundability level λ

we write

$$h(t|\lambda) = \lambda e^{-\delta t} \quad \text{where } \delta > 0$$

as the conditional hazard rate.

Assuming the variation of fecundability between women to women conforming to a Gamma distribution with two parameters, following Brass (1958) we get

$$\phi(\lambda) = \frac{e^{-a\lambda} \lambda^{k-1} a^k}{\Gamma(k)} \quad \ldots (3.3)$$

$$a, k > 0$$

$$0 \leq \lambda < \infty$$

we have the unconditional hazard rate

$$\begin{aligned} h(t) &= \int_0^\infty h(t|\lambda)\phi(\lambda)d\lambda \\ &= \frac{a^k e^{-\delta t}}{\Gamma(k)} \int_0^\infty \lambda^k e^{-a\lambda} d\lambda \\ &= \frac{k}{a} e^{-\delta t} \quad \ldots (3.4) \end{aligned}$$

where t is age of the woman.

Also

$$V_0(t|\lambda) = \text{Prob(no birth upto time } t)$$

$$= e^{-\int_0^t h(\tau|\lambda)\,d\tau}$$

$$= e^{-\lambda\left[\frac{1}{\delta}(1-e^{-\delta t})\right]} = e^{-\lambda A(t)} \quad \ldots (3.5)$$

where $$A(t) = \left[\frac{1}{\delta}(1-e^{-\delta t})\right] \quad \ldots (3.6)$$

$$\Rightarrow \quad V_0(t) = \int_0^t V_0(t|\lambda)\,\phi(\lambda)\,d\lambda$$

$$= \frac{a^k}{(a+A(t))^k} \quad \ldots (3.7)$$

It may be seen that

$$\lim_{t \to 0} \frac{1-V_0(t)}{t} = \frac{k}{a}$$

which is independent of t showing the stationarity of the process.

By differentiating Palm's integral equation (3.2) $\Rightarrow$

$$\frac{d}{dt} V_0(t) = -\frac{k}{a}\,\phi_0(t)$$

$$\Rightarrow \phi_0(t) = \frac{a^{k+1}}{(a+A(t))^{k+1}}\,A'(t) \quad \ldots .(3.8)$$

where $$A'(t) = e^{-\delta t} \quad \ldots ..(3.8')$$

$$\therefore \quad \phi_0(t) = \frac{a^{k+1}}{\left(a + \frac{1}{\delta}(1 - e^{-\delta t})\right)} e^{-\delta t} \qquad \cdots (3.9)$$

Since $(V_n(t) \mid \lambda)$ conforms to a differential difference equation given by $n = 0,1,2,\ldots\ldots$

we have

$$V_n(t+\delta t \mid \lambda) = V_{n-1}(t)\left[h(t \mid \lambda)\, e^{-\delta t} + o(\delta t)\right]$$

$$+ V_n(t)\left[1 - \left[h(t \mid \lambda) e^{-\delta t} + o(\delta t)\right]\right] \qquad \cdots (3.10)$$

$$\Rightarrow \frac{d}{dt} V_n(t \mid \lambda) = \lambda\, e^{-\delta t}\left[V_{n-1}(t \mid \lambda) - V_n(t \mid \lambda)\right]$$

We have

$$V_n(t \mid \lambda) = \frac{\left(\lambda \int_0^t e^{-\delta t}\, dt\right)^n}{n!}\, e^{-\lambda \int_0^t e^{-\delta t} dt} \qquad \cdots (3.11)$$

Again by Palm's integral equation,

for $k = 1$,

$$\frac{d}{dt} V_1(t \mid \lambda) = \lambda\, e^{-\delta t}\left(V_0(t \mid \lambda) - V_1(t \mid \lambda)\right)$$

$$\Rightarrow V_1(t \mid \lambda) = \frac{k}{a} A(t)\left[\frac{a}{a + A(t)}\right]^{k+1} \qquad \cdots (3.12)$$

Where $A(t)$ is given in (3.6)

$$\Rightarrow \phi_1(t) = \phi_0(t) - \frac{a}{k}\frac{d}{dt} V_1(t)$$

$$= \frac{(k+1)\, a^{k+1} A(t)\, A'(t)}{1!\, (a + A(t))^{k+2}} \qquad \ldots(3.13)$$

Precisely in a similar way

$$V_2(t) = \frac{k(k+1)(A(t))^2 a^k}{2!\, (a+A(t))^{k+2}}$$

$$\Rightarrow \phi_2(t) = \frac{(k+1)(k+2)\, a^{k+1} A'(t)(A(t))^2}{2!\, (a + A(t))^{k+3}} \qquad \ldots(3.14)$$

$$V_3(t) = \frac{k(k+1)(k+2)\, a^k (A(t))^3}{3!\, (a + A(t))^{k+3}} \qquad \ldots(3.15)$$

$$\Rightarrow \phi_3(t) = \frac{(k+1)(k+2)(k+3)\, a^{k+1} A'(t)(A(t))^3}{3!\, (a + A(t))^{k+4}} \qquad \ldots(3.16)$$

Proceeding in this way we get

$$\phi_n(t) = \frac{(k-1)(k+2).......(k+n)\, a^{k+1} A'(t)(A(t))^n}{n!\, (a+A(t))^{n}} \qquad .....(3.17)$$

$$n = 1,2,3,........$$

Expected number of births in the residual fertility span $(T-t_1)$ given that the first birth has occurred at $t=t_1$ for the first cohort is

$$\sum_{n=0}^{\infty} n\, \phi_n^{(1)}(T-t_1)$$

$$= \left(1 - \frac{A(T-t_1)}{a_1 + A(T-t_1)}\right)^{-(K_1+2)} \frac{a_1^{k_1+1} A(T-t_1)(K_1+1)A'(T-t_1)}{(a_1 + A(T-t_1))}$$

$$= f_1(a_1, k_1, \delta, (T-t_1)) \qquad \ldots(3.18)$$

and the expected number of births in the residual fertility span, given that the first birth has occurred at $t=t_2$, for the second cohort is

$$\sum_{n=0}^{\infty} n\, \phi_n^{(2)}(T-t_2)$$

$$= \left(1-\frac{A(T-t_2)}{a_2+A(T-t_2)}\right)^{-(k_2+2)} \frac{a_2^{k_2+1} A(T-t_2)(k_2+1)A'(T-t_2)}{(a_2+A(T-t_2))}$$

$$= f_2(a_2, k_2, \delta, (T-t_2)) \qquad \ldots (3.19)$$

where (a_i, k_i), $i = 1,2$ are parameters of the Gamma distribution corresponding to the cohorts I and II respectively.

Therefore for a change of age at marriage by

$= t_2 - t_1$ $\qquad t_2 > t_1$

there is change in the average level of births

$$E = \begin{bmatrix} f_1(a_1, k_1, T-t_1, s) \\ \qquad -f_2(a_2, k_2, T-t_2, s) \end{bmatrix} \qquad \ldots (3.20)$$

$f_1 > f_2$ indicates fall in the overall fertility level

$f_1 < f_2$ indicates increase in the fertility level; and

$f_1 \simeq f_2$ indicates the stationarity in the fertility level.

Estimation of the Parameters of the Model:

We have for the first cohort $E_1[X(T-t_1)|$ there is a birth at $t = t_1]$

$$= \left(1-\frac{A(T-t_1)}{a_1+A(T-t_1)}\right)^{-(k_1+2)} \frac{a_1^{k_1+1} A(T-t_1)(k_1+1)A'(T-t_1)}{(a_1+A(T-t_1))}$$

$$= f_1(a_1,k_1,\delta,T-t_1) \text{ say} \qquad (3.18)$$

and for the second cohort

$E_2(X(T-t_2)|$ there is a birth at $t=t_2)$

$$= \left(1-\frac{A(T-t_2)}{a_2+A(T-t_2)}\right)^{-(k_2+2)} \frac{a_2^{k_2+1} A(T-t_2)(k_2+1)A'(T-t_2)}{(a_2+A(T-t_2))}$$

$$= f_2(a_2,k_2,\delta,T-t_2) \text{ say} \qquad (3.19)$$

Then the variance of the number of births for the first cohort

$$\mathrm{Var}(X(T-t_1)|\text{birth at } t=t_1) = \sum_{n=0}^{\infty} n^2 \phi_n^{(1)}(T-t_1) - \left[\sum_{n=0}^{\infty} n\phi_n^{(1)}(T-t_1)\right]^2$$

$$= \frac{(k_1+1)a_1^{k_1+1}A'(T-t_1)A(T-t_1)}{a_1+A(T-t_1)}\left(\frac{a_1}{a_1+A(T-t_1)}\right)^{-(k_1+2)}$$

$$+ a_1^{k_1+1}A'(T-t_1)(k_1+1)(k_1+2)\left(\frac{A(T-t_1)}{a_1+A(T-t_1)}\right)^2\left(\frac{a_1}{a_1+A(T-t_1)}\right)^{-(k_1+3)}$$

$$-\left[\frac{a_1^{k_1+1}A(T-t_1)(k_1+1)A'(T-t_1)}{(a_1+A(T-t_1))}\left(\frac{a_1}{a_1+A(T-t_1)}\right)^{-(k_1+2)}\right]^2$$

$$= f_3(a_1,k_1,\delta_1,T-t_1) \text{ say} \qquad \ldots \qquad (3.21)$$

and the variance of the number of births corresponding to the second cohort

$$= \sum_{n=0}^{\infty} n^2 \phi_n^{(2)}(T-t_2) - \left[\sum_{n=0}^{\infty} n\phi_n^{(2)}(T-t_2)\right]^2$$

$$= \frac{(k_2+1)a_2^{k_2+1} A'(T-t_2)A(T-t_2)}{(a_2+A(T-t_2))} \cdot \left(\frac{a_2}{a_2+A\,(T-t_2)}\right)^{-(k_2+1)} .$$

$$+a_2^{k_2+1} A'(T-t_2)(k_2+1)(k_2+2) \left(\frac{A(T-t_2)}{a_1+A(T-t_2)}\right)^2 \left(\frac{a_2}{a_2+A\,(T-t_2)}\right)^{-(k_2+3)}$$

$$-\left[a_2^{k_2+1} \frac{A(T-t_2)(k_2+1)A'(T-t_2)}{(a_2+A(T-t_2))} \left(\frac{a_2}{a_2+A\,(T-t_2)}\right)^{-(k_2+2)}\right]^2$$

$$= \quad f_4(a_2, k_2, \delta_2, T-t_2) \quad \text{say} \qquad ..(3.22)$$

Now by the method of moments (3.18) and (3.21) for given δ and $T-t_1$ will enable us to estimate a_1 and k_1 by successive approximations.

Similarly by the same procedure (3.19) and (3.22) for given δ and $T-t_2$ the estimates of a_2 and k_2 will be obtained by successive iterations.

On substitution of the estimates $\hat{a}_1$, $\hat{k}_1$, $\hat{a}_2$, $\hat{k}_2$, and $\hat{\delta}$ for given $(T-t_1)$ and $(T-t_2)$ one can examine the positiveness (or negativeness) of the expression in (3.20) to ascertain the increasing or decreasing trend of fertility as observed in two cohorts who differ by τ years at age at marriage. The inherent difference in the fertility, apart

from the availability of total marital exposure, if any, is expressed in the estimated values of the parameters (a_i, k_i) $i = 1,2$, for two cohorts.

A Numerical Illustration

Let

$$E_1(X(T-t_1) \mid (t_1) = f_1(a_1, k_1, \delta, T-t_1) = (A) : 3.64, 4.48, 5.50$$

$$Var_1(X(T-t_1) \mid t_1) = f_3(a_1, k_1, \delta, T-t_1) = (B) = 10.49, 9.65, 10.69$$

and $T-t_1 = 45-16 = 29$ years

and

$$E_2(X(T-t_2) \mid t_2) = f_2(a_2, k_2, \delta, T-t_2) = (C) = 3.09, 4.40, 4.62$$

and

$$Var_2(X(T-t_2) \mid t_2) = f_4(a_2, k_2, \delta, T-t_2) = (D) = 9.43, 15.20, 7.90$$

and $T-t_2 = 45-18 = 27$ years.

Solving (A) and (B) for the given set of hypothetical values as given above by starting with an approximate initial solution of $a_1 = a_1^o$ and $k_1 = k_1^o$, we get by successive iteration

$$\hat{a}_1 = 0.35, 0.40, 0.40, \quad \hat{k}_1 = 1.0, 1.5, 2.0$$

Similarly starting with an approximate initial solution of $a_2 = a_2^o$ and $k_2 = k_2^o$ we solve (C) and (D) by successive iterations and get

$$a_2 = 0.35, 0.35, 0.45 \quad k_2 = 1.0, 2.0, 2.0.$$

Substituting the estimates of a_1, k_1 and a_2, k_2 in (3.20) we get three values of

$$E = 0.27,\ -0.24,\ 0.48 \text{ respectively}$$

which shows that in the hypothetical exercise the first cohort has the fertility status not significantly different than that of the second one.

<u>A note on the nature of the Palm Probability Distribution of the number of births during fixed marital exposure</u> .

A comparison of the mean and the variance of the Palm Probability distribution given in (3.17) shows that

$$\text{Variance} = \text{Mean} + c_i\ (\text{Mean}) - (\text{mean})^2$$

where

$$c_i = (k+3)\, \frac{\frac{1}{\delta}\left(1 - e^{-\delta(T-t_i)}\right)}{\left[a + \frac{1}{\delta}\left(1 - e^{-\delta(T-t_1)}\right)\right]} \left(\frac{a}{\left[a + \frac{1}{\delta}\left(1 - e^{-\delta(T-t_1)}\right)\right]} \right)^{-1}$$

$$= \left[\left(\frac{k+2}{a}\right) \frac{1}{\delta}\left(1 - e^{-\delta\left(1 - e^{-\delta(T-t_i)}\right)}\right)\right], \quad i = 1,2$$

If k, δ and a satisfy a relation so that $c_i = \theta$ (mean).

For the non-negativity of the variance it is necessary that $\theta \geq 1$ for Mean $>$ 1.

$\Rightarrow$ Variance = Mean $(1 + (c_i - (\text{mean}))$

Variance = Mean $(1 + (\text{mean}\ (\theta - 1))$.

In particular, it may be noted that if mean is greater than one, if $0 \leq \theta < 1$, Variance $<$ Mean; or $\theta = 1 \Rightarrow$ Variance = Mean.

$0 < \theta \Rightarrow$ Variance $>$ Mean. For $\theta = 2$, Variance = Mean = $(\text{Mean})^2$.

Hence the model is flexible so that by suitable choice of the parameters the mean/variance ratio can be made in the same order as may be observed in the sample.

CHAPTER-4

ON THE APPLICATION OF PALM PROBABILITY FOR OBTAINING THE WAITING TIME DISTRIBUTION BETWEEN THE FIRST AND THE HIGHER ORDER OF CONCEPTIONS

Singh (1964) has modelled a probability distribution on the time of first birth. Assuming that a conception taking place during the fecundable period following the marriage with hazard rate λ , the waiting time distribution for the first conception is given by

$$f(t) = \lambda e^{-\lambda t} \qquad 0 \leq t < \infty \qquad \cdots(4.1)$$

On further assumption, that the fecundability parameter λ varies from individual to individual (even with the same parity group) following a probability density given by

$$\psi(\lambda) = \frac{a^k}{\Gamma(k)} e^{-a\lambda} \lambda^{k-1} \qquad \cdots(4.2)$$
$$a, k > 0$$

the waiting time distribution for the first conception following the effective marriage is modelled by Singh (1964) as

$$\psi(t) = \frac{K a^k}{(a+t)^{k+1}} , \quad a, k > 0 \qquad (4.3)$$

However, the problem of obtaining the waiting time distribution between the first and the second conception or between the

first and the nth conception (n=2,3,.......) cannot be obtained as a direct generalisation of the result obtained by Singh (1964); because on account of the weighting of the Poisson Process by a Gamma distribution the renewal structure of the process is completely destroyed leading to inter-arrival distributions having infinitely divisible structures with dependent increments. (Biswas and Pachal (1983)).

Denoting by X(t) the number of compound Poisson events (weighted by a Gamma distribution given by (4.2) it follows that for all $t > s$

$$\begin{aligned} & \mathrm{Cov}\,(X(s),\ X(t) - (s)) \\ = \quad & E_{\lambda}\,(X(s)\ (X(t) - X(s) \mid \lambda\ ) - [E\ (\lambda)]^2\ S(t-S) \\ = \quad & E_{\lambda}\,(\ \lambda^{2}\ s(t-s)) - (E(\ \lambda\ ))^2\ \ s(t-s) \\ = \quad & s(t-s)\ ((E\ \ \lambda^{2})) - (E(\ \ \lambda\ )^2) = s(t-s)\ \mathrm{Var}\ (\ \lambda\ ). \end{aligned}$$

... (4.4)

This makes the problem of obtaining the probability distribution of the inter-arrival in a compound Poisson process (or any such compound dependent process) more complicated. On the other hand, this is the type of distribution we require while obtaining the waiting time distribution between the first and the second conceptions. Using the concept of Palm Probability (Khinchine (1960)), Biswas and Pachal (1983) have however obtained the probability distribution of the Ist and the nth conception (n=2,3,.....) given that the first conception has occured at time T=0.

$$f_n(t|\cdot) = \frac{t^{n-1}\,\alpha^{k+1}\,(k+1)(k+2)\cdots(k+n)}{(n-1)!\,(a+t)^{k+n+1}} \quad \ldots (4.5)$$

which is based on a similar assumption of having a time independent intensity of conception and assuming to conform to a distribution as in (3.2). The results have been proved in earlier chapter (vide Chapter two).

For n=1 i.e the probability of the waiting time for second conception subje t to the first conception taking place at T=0, is therefore, given by

$$f_1(t \mid \cdot \ ) = \frac{a^{k+1} \quad (k+1)}{(a+t)^{k+2}} \qquad \text{....(4.6)}$$

It may be noted that the probability distribution of the waiting time between the effective marriage and the first complete conception (given by (4.3) is obtainable from the corresponding distribution of the waiting time between the first and the second conception given by (4.6) replacing the parameter K+1 to K and vice versa.

However there are several ways in which the model (4.5) can be generalised. As in Chapter 3 if we take $h(t|\lambda) = \lambda e^{-\delta t}$, $\delta > 0$ instead of $h(t \mid \lambda) = \lambda$ based on which (4.5) is developed, we have by the same notation as in Chapter 3.

$G(z,t)$ = Probability generating function of $V_\gamma(t)$,

$\gamma = 1, 2, \ldots\ldots$

$$= \sum_{\gamma=0}^{\infty} z^{\gamma} V_{\gamma}(t)$$

$$= \sum_{\gamma=0}^{\infty} z^{r} \int_{0}^{\infty} V_{\gamma}(t|\lambda)\phi(\lambda)d\lambda$$

where

$$V_{\gamma}(t|\lambda) = \frac{\left(\lambda \int_{0}^{t} e^{-\delta t} dt\right)^{\gamma}}{\gamma!} e^{-\lambda \int_{0}^{t} e^{-\delta t} dt}$$

$$= \frac{\lambda^{\gamma}}{\gamma!} \left(\frac{1 - e^{-\delta t}}{\delta}\right)^{\gamma} e^{-\lambda\left(\frac{1-e^{-\delta t}}{\delta}\right)}$$

$$= \frac{[\lambda A(t)]^{\gamma}}{\gamma!} e^{-\lambda A(t)}$$

Putting $\dfrac{1-e^{-\delta t}}{\delta} = A(t)$

$$G(z,t) = \sum_{\gamma=0}^{\infty} z^{\gamma} \int_{0}^{\infty} \frac{[\lambda A(t)]^{\gamma}}{\gamma!} e^{-\lambda A(t)} \frac{e^{-a\lambda} \lambda^{K-1} a^{k}}{\Gamma(K)} d\lambda$$

$$= \sum_{\gamma=0}^{\infty} z^{\gamma} \frac{(A(t))^{\gamma} a^{k}}{\gamma!\, \Gamma(K)} \int_{0}^{\infty} e^{-\lambda[a + A(t)]} \lambda^{\gamma+k-1} d\lambda$$

$$= \sum_{\gamma=0}^{\infty} z^{\gamma} \frac{(A(t))^{\gamma} a^{k}}{\gamma\,!\,\Gamma(k)} \frac{\Gamma(\gamma+k)}{(a+A(t))^{\gamma+k}}$$

$$= \frac{a^{k}}{\Gamma(k)} \sum_{\gamma=0}^{\infty} \frac{z^{\gamma}(A(t))^{\gamma}\Gamma(\gamma+k)}{\gamma!(a+A(t))^{\gamma+k}}$$

$$= \frac{a^{k}}{\Gamma(k)} \frac{1}{(a+A(t))^{k}} \sum_{\gamma=0}^{\infty} \left(\frac{zA(t)}{a+A(t)}\right)^{\gamma} \frac{\Gamma(\gamma+k)}{\gamma!}$$

$$= \frac{1}{\Gamma(k)} \left(\frac{a}{a+A(t)}\right)^{k} \sum_{\gamma=0}^{\infty} \left(\frac{zA(t)}{a+A(t)}\right)^{\gamma} \frac{\Gamma(\gamma+k)}{\gamma!}$$

$$= \frac{1}{\Gamma(k)} \left(\frac{a}{a+A(t)}\right)^{k} \left(\Gamma(k) + \frac{\Gamma(k+1)}{1!} \frac{zA(t)}{a+A(t)} + \frac{\Gamma(k+2)}{2!} \left(\frac{zA(t)}{a+A(t)}\right)^{2} + \frac{\Gamma(k+3)}{3!} \left(\frac{zA(t)}{a-A(t)}\right)^{3} + \ldots\ldots\ldots\right)$$

$$= \frac{1}{\Gamma(k)} \left(\frac{a}{a+A(t)}\right)^{k} \left((k-1)! + k!\, \frac{zA(t)}{a-A(t)} + \frac{(k+1)!}{2!} \left(\frac{zA(t)}{a+A(t)}\right)^{2} + \frac{(k+2)!}{3!} \left(\frac{zA(t)}{a+A(t)}\right)^{3} + \ldots\ldots\ldots\right)$$

$$= \frac{1}{\Gamma(k)} \left(\frac{a}{a+A(t)}\right)^{k} \left((k-1)!\,\left(1+ \frac{kzA(t)}{a+A(t)}\right.\right.$$

$$+ \frac{k(k+1)}{2!}\left(\frac{zA(t)}{a+A(t)}\right)^2 + \frac{k(k+1)(k+2)}{3!}\left(\frac{zA(t)}{a+A(t)}\right)^3 + \ldots\ldots\ldots))$$

$$= \left(\frac{a}{a+A(t)}\right)^k \left(1+ \frac{kzA(t)}{a+A(t)} + \frac{k(k+1)}{2!}\left(\frac{zA(t)}{a+A(t)}\right)^2 + \frac{k(k+1)+(k+2)}{3!}\left(\frac{zA(t)}{a+A(t)}\right)^3 + \ldots\ldots\right)$$

$$= \left(\frac{a}{a+A(t)}\right)^k \left(1 - \frac{zA(t)}{z+A(t)}\right)^{-k}$$

$$= \left(\frac{a}{a+A(t)}\right)^k \cdot \left\{\frac{a+A(t)}{a+A(t)-zA(t)}\right\}^K$$

$$= \left(\frac{a}{a+A(t)(1-z)}\right)^k \qquad \ldots.(4.6)$$

($\gamma_\gamma(t)$, $\phi(t)$, $A(t)$ are same as defined in Chapter 3). Again using a result of Cox and Isham (1980)

$$\frac{\partial}{\partial t} G(z,t) = \frac{-k}{a}(1-z)G_0(z,t) \qquad \ldots\ldots(4.7)$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{a}{a+A(t)(1-z)}\right)^k = \frac{-k}{a}(1-z)G_0(z,t)$$

$$\Rightarrow a^k(-k(a+A(t)(1-z))]^{k-1}(1-z)A'(t) = \frac{-k}{a}(1-z)G_0(z,t)$$

$$\Rightarrow a^{k+1}(a+A(t)(1-z))^{-(k+1)}A'(t) = G_0(z,t)$$

$$\Rightarrow G_0(z,t) = \left(\frac{a}{a+A(t)(1-z)}\right)^{k+1} A'(t)$$

$= \text{p.g.f. of } \phi_k(t)$

$$= \sum_{k=0}^{\infty} z^k \phi_k(t). \qquad \ldots..(4.8)$$

Denoting by $\widetilde{F}_n(t) = P[T_n \leq t \,/\,$ there is an event at $t=0]$

Where T_n is the conditional waiting time (a random variable) for the nth arrival following one arrival at $t=0$. We have

$$\widetilde{F}_n(t) = \sum_{N=n}^{\infty} \phi_N(t) \qquad \text{....(4.9)}$$

and

$$H_o(z,t) = \sum_{n=0}^{\infty} z^n \widetilde{F}_n(t) \qquad \text{....(4.10)}$$

$$\Leftrightarrow \quad \frac{\partial}{\partial t} H_o(z,t) = \sum_{n=0}^{\infty} z^n \widetilde{f}_n(t). \qquad \text{...(4.11)}$$

We have

$$H_o(z,t) = \frac{1 - z\,G_o(z,t)}{1-z} \qquad \text{...(4.12)}$$

(4.10), (4.11) and (4.12) $\Rightarrow$

$$-\frac{z}{1-z}\frac{\partial}{\partial t} G_o(z,t) = \sum_{n=0}^{\infty} z^n \widetilde{f}_n(t) \qquad \text{.....(4.13)}$$

Again (4.8) $\Rightarrow$

$$\frac{\partial}{\partial t} G_o(z,t) = a^{k+1}\Big[-(K+1)[a + A(t)(1-z)]^{-(k+2)} (1-z)(A'(t))^2 + [a + A(t)(1-z)]^{-(k+1)} A''(t)\Big]$$

$$= a^{k+1}\left[\frac{A''(t)}{[a + A(t)(1-z)]^{k+1}} - \frac{(k+1)(1-z)(A'(t))^2}{(a+A(t)(1-z))^{k+2}}\right]$$

$$= \frac{a^{k+1}}{[a + A(t)(1-z)]^{k+1}}\left[A''(t) - \frac{(k+1)(1-z)(A'(t))^2}{a + A(t)(1-z)}\right] \qquad \text{.....(4.14)}$$

(4.7) and (4.8) $\Rightarrow$

$$\sum_{n=0}^{\infty} z^n \tilde{f}_n(t) = -\frac{z}{1-z}\left(\frac{a}{a+A(t)(1-z)}\right)^{k+1}$$

$$\left(A''(t)\right) - \frac{(k+1)(1-z)(A'(t))^2}{a+A(t)(1-z)}) \qquad ..(4.15)$$

$$= -\frac{z}{1-z}\left(\frac{a}{a+A(t)(1-z)}\right)^{k+1} A''(t)$$

$$+ \frac{z(k+1)a^{k+1}(A'(t))^2}{\left[a+A(t)(1-z)\right]^{k+2}}$$

$$= -\frac{z}{1-z}\left(\frac{z}{a'-zA(t)}\right)^{k+1} A''(t)$$

$$+ \frac{z(k+1)a^{k+1}(A'(t))^2}{\left[a'-zA(t)\right]^{k+2}}$$

where $a' = a+A(t)$

$$= -\frac{z}{1-z}\left(\frac{1}{\frac{a'}{a}-\frac{zA(t)}{a}}\right)^{k+1} A''(t) + \frac{z(k+1)}{a}\frac{(A'(t))^2}{\left[\frac{a'}{a}-\frac{zA(t)}{a}\right]^{k+2}}$$

$$= -\frac{zA''(t)}{1-z}\left(\frac{a'}{a}-\frac{zA(t)}{a}\right)^{-(k+1)}$$

$$+ \frac{z(k+1)}{a}(A'(t))^2\left[\frac{a'}{a}-\frac{zA(t)}{a}\right]^{-(k+2)}$$

$$= -\frac{zA''(t)}{1-z}\left(\frac{a'}{a}\right)^{-(k+1)}\left[1-\frac{zA(t)}{a}\right]^{-(k+1)}$$

$$+ \frac{z(k+1)}{a}(A'(t))^2\left(\frac{a'}{a}\right)^{-(k+2)}\left[1-\frac{zA(t)}{\alpha'}\right]^{-(k+2)}$$

$$= -\frac{zA''(t)}{1-z}\left(\frac{a'}{a}\right)^{-(k+1)}\left[1+\frac{(k+1)\,zA(t)}{a'}\right.$$

$$+ \frac{(k+1)(k+2)}{1.2}\left(\frac{zA(t)}{a'}\right)^2 + \cdots$$

$$\left. + \frac{(k+1)(k+2)\ldots\ldots.(k+n)}{1.2\ldots\ldots..n}\left(\frac{zA(t)}{a'}\right)^n + \cdots\right]$$

$$+ \frac{z(k+1)(A'(t))^2}{a}\left(\frac{a'}{a}\right)^{-(k+2)}\left[1+(k+2)\frac{zA(t)}{a'}\right.$$

$$+\frac{(k+2)(k+3)}{1.2}\left(\frac{zA(t)}{a}\right)^2 + \cdots$$

$$+\frac{(k+2)(k+3)\ldots\ldots(k+n)}{1.2\ldots\ldots(n-1)}\left(\frac{zA(t)}{a}\right)^{n-1} + \cdots\cdots\Big]$$

$$= \quad -z(1+z+z^2+z^3+\ldots\ldots\ldots\ldots+z^{n-1}+z^n=\ldots\ldots)A''(t)\left(\frac{a'}{a}\right)^{-(k+1)}$$

$$\Big[1+\frac{(k+1)A(t)z}{a'} + \frac{(k+1)(k+2)}{1.2}\left(\frac{zA(t)}{a'}\right)^2 + \cdots$$

$$+ \frac{(k+1)(k+2)\ldots\ldots(k+n-1)}{1.2\ldots\ldots(n-1)}\left(\frac{zA(t)}{a'}\right)^{n-1} + \cdots\cdots$$

$$+ z(k+1)\frac{(A'(t))^2}{a}\left(\frac{a'}{a}\right)^{-(k+2)}\Big[1+\frac{(k+2)zA(t)}{a'}$$

$$+ \frac{(k+2)(k+3)}{1.2}\left(\frac{zA(t)}{a'}\right)^2 + \cdots$$

$$+ \frac{(k+2)(k+3)\ldots\ldots\ldots(k+n)}{1.2\ldots\ldots(n-1)}\left(\frac{zA(t)}{a'}\right)^{n-1} + \cdots\cdots\Big]$$

$$= \quad -(z+z^2+z^3+\ldots\ldots+z^n+\ldots\ldots)A''(t)\left(\frac{a'}{a}\right)^{-(k+1)}$$

$$\Big[1+\frac{(k+1)zA(t)}{a'} + \frac{(k+1)(k+2)}{1.2}\left(\frac{zA(t)}{a'}\right)^2 + \cdots$$

$$+ \frac{(k+1)(k+2)\ldots\ldots(k+n-1)}{1.2\ldots(n-1)}\left(\frac{zA(t)}{a'}\right)^{n-1} + \cdots\cdots\Big]$$

$$+ \quad z(k+1)\frac{(A'(t))^2}{a}\left(\frac{a'}{a}\right)^{-(k+2)}\Big[1+\frac{(k+2)zA(t)}{a'}$$

$$+ \frac{(k+2)(k+3)}{1.2}\left(\frac{zA(t)}{a'}\right)^2 + \cdots\cdots\cdots\cdots$$

$$+ \frac{(k+2)\ldots\ldots\ldots(k+n)}{1.2\ldots\ldots\ldots(n-1)}\left(\frac{zA(t)}{a'}\right)^{n-1} + \cdots\cdots\Big] \qquad (4.16)$$

Equating the coefficients of z^n on both sides, we get

$$\tilde{f}_n(t) = -A''(t)\left(\frac{a'}{a}\right)^{-(k+1)}\Big[1+\frac{(k+1)A(t)}{a'}$$

$$+\frac{(k+1)(k+2)}{1.2}\left(\frac{A(t)}{a'}\right)^2+\dots+\frac{(k+1)(k+2)\dots(k+n-1)}{1.2\dots\dots(n-1)}\left(\frac{A(t)}{a'}\right)^{n-1}\Big]$$

$$+\frac{(k+1)(A'(t))^2}{a}\left(\frac{a'}{a}\right)^{-(k+2)}\left(\frac{(k+2)\dots(k+n)}{1.2\dots(n-1)}\left(\frac{A(t)}{a'}\right)\right)$$

$$= \frac{a^{k+1}}{a'^{k+1}}\,\delta\, e^{-\delta t}\Big[1+(k+1)\frac{A(t)}{a'}+\frac{(k+1)(k+2)}{1.2}\left(\frac{A(t)}{a'}\right)^2+\dots\dots\dots$$

$$\dots\dots+\frac{(k+1)\dots\dots(k+n-1)}{1.2\dots\dots(n-1)}\left(\frac{A(t)}{a'}\right)^{n-1}\Big]$$

$$+\frac{(k+1)a^{k+1}}{a'^{k+2}}e^{-2\delta t}\Big[\frac{(k+2)\dots\dots(k+n)}{1.2\dots\dots(n-1)}e\left(\frac{A(t)}{a'}\right)^{n-1}\Big]$$

$$= \frac{a^{k+1}e^{-\delta t}}{(a+A(t))^{k+1}}\Big[\delta+\frac{(k+1)\delta A(t)}{a+A(t)}+\frac{(k+1)(k+2)\delta}{1.2}\left(\frac{A(t)}{a+A(t)}\right)^2+\dots$$

$$+\frac{(k+1)\dots(k+n-1)\delta}{1.2\dots\dots(n-1)}\left[\frac{A(t)}{a+A(t)}\right]^{n-1}+\frac{(k+1)\dots(k+n)}{1.2\dots(n-1)}e^{-\delta t}\frac{(A(t))^{n-1}}{(a+A(t))^n}\Big]$$

$$= \frac{a^{k+1}e^{-\delta t}}{[a+A(t)]^{k+1}}\Big(\delta+\frac{(k+1)\delta A(t)}{a+A(t)}+\frac{(k+1)(k+2)\delta}{2!}\left[\frac{A(t)}{a+A(t)}\right]^2+\dots$$

$$+\frac{(k+1)(k+2)\dots\dots(k+n-1)\delta}{(n-1)!}\left(\frac{A(t)}{a+A(t)}\right)^{n-1}$$

$$+\frac{(k+1)\dots\dots(k+n)}{(n-1)}e^{-\delta t}\frac{(A(t))^{n-1}}{(a+A(t))^n}\Big] \qquad \dots(4.17)$$

Again

$$\tilde{F}_n(t) = \sum_{N=n}^{\infty}\phi_N(t)$$

$$= 1-\Big(\frac{a^{k+1}A'(t)}{(a+A(t))^{k+1}}+\frac{(k+1)a^{k+1}A'(t)A(t)}{(a+A(t))^{k+2}}$$

$$+\frac{(k+1)(k+2)a^{k+1}A'(t)(A(t))^2}{2!(a+A(t))^{k+3}}+\dots\dots\dots$$

$$+\frac{(k+1)(k+2)\dots\dots(k+n-1)a^{k+1}A'(t)(A(t))^{n-1}}{(n-1)!(a+A(t))^{k+n}}\Big) \qquad \dots(4.18)$$

Using $E_n(T_n|t_0) = t_0 + \int_0^{\infty} [1 - \tilde{F_n}(t)]\,dt = \int_0^{\infty} \tilde{R_n}(t)\,dt$ ....(4.19)

where $\tilde{R}_n(t)$ is the survival function corresponding to the c.d.f. $\tilde{F}_n(t)$.

Putting n = 1,2,3,... and by straight forward integration we have

$$E_1(T_1|t_0) = t_0 + \int_0^{\infty} \frac{a^{k+1} A'(t)}{(a+A(t))^{k+1}}\,dt$$

$$= t_0 + a^{k+1} \int_a^{a+\frac{1}{\delta}} z^{-(k+1)}\,dz$$

where $a+A(t) = z$

$$\Rightarrow A'(t)dt = dz$$

$$\Rightarrow t_0 + a^{k+1} \frac{z^{-k}}{-k}\Big|_a^{a+1}$$

$$= t_0 + a^{k+1} \left[\frac{1}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)\right]$$

$$= f_1(a,k,\delta), \text{ say} \qquad \text{....(4.20)}$$

$$E_2(T_2|t_0) = t_0 + \int_0^{\infty} \left(\frac{a^{k+1}A'(t)}{[a+A(t)]^{k+1}} + \frac{(k+1)a^{k+1}A'(t)A(t)}{[a+A(t)]^{k+2}}\right)dt$$

$$= t_0 + a^{k+1} \int_0^{\infty} \frac{A'(t)}{[a+A(t)]^{k+1}}\,dt$$

$$+ (k+1)a^{k+1} \int_0^{\infty} \frac{A'(t)A(t)}{[a+A(t)]^{k+2}}\,dt$$

$$= t_0 + \frac{a^{k+1}}{k}\left[\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + (k+1)a^{k+1} \int_a^{a+\frac{1}{\delta}} (z-a)z^{-(k+2)}\,dz$$

where $a+A(t) = z$

$\Rightarrow A'(t)dt = dz$

$$= t_o + a^{k+1}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + (k+1)a^{k+1}\left[\frac{z^{-k}}{-k}\Big|_a^{a+\frac{1}{\delta}} - a\,\frac{z^{-(k+1)}}{-(k+1)}\Big|_a^{a+1/\delta}\right)$$

$$= t_o + \frac{a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + \frac{(k+1)a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)$$

$$-\frac{(k+1)a^{k+2}}{k+1}\left(\frac{1}{a^{k+1}} - \frac{1}{(a+\frac{1}{\delta})^{k+1}}\right)$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + a^{k+1}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)$$

$$- a^{k+2}\left(\frac{1}{a^{k+1}} - \frac{1}{(a+\frac{1}{\delta})^{k+1}}\right)$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$= f_2(a,k,\delta), \text{ say} \qquad \ldots(4.21)$$

$$E_3(T_3|t_o) = t_o + \int_0^\infty\left(\frac{a^{k+1}A'(t)}{(a+A(t))^{k+1}} + \frac{(k+1)a^{k+1}A'(t)A(t)}{(a+A(t))^{k+2}}\right.$$

$$\left. + \frac{(k+1)(k+2)a^{k+1}}{2!}\,\frac{A'(t)\,(A(t))^2}{(a+A(t))^{k+3}}\right)dt$$

$$= t_o + a^{k+1}\int_0^\infty \frac{A'(t)}{(a+A(t))^{k+1}}dt + (k+1)a^{k+1}\int_0^\infty \frac{A'(t)A(t)}{(a+A(t))^{k+2}}\,dt$$

$$+ \frac{(k+1)(k+2)a^{k+1}}{2!}\int_0^\infty \frac{A'(t)(A(t))^2}{(a+A(t))^{k+3}}dt$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{(k+1)(k+2)a^{k+1}}{2}\, a \int_{d}^{a+\frac{1}{\delta}} (z-a)^2 z^{-(k+3)} dz$$

where $z = a+A(t)$

$\Rightarrow$ $dz = A'(t)dt$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{(k+1)(k+2)a^{k+1}}{2} \int_{a}^{a+\frac{1}{\delta}} (z^2+a^2-2az) z^{-(k+3)} dz$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{(k+1)(k+2)a^{k+1}}{2}\left(\frac{z^{-k}}{-k} + a^2\frac{z^{-(k+2)}}{-(k+2)} - \frac{2az^{-(k+1)}}{-(k+1)} \Big|_{a}^{a+\frac{1}{\delta}}\right)$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a-\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{(k+1)(k+2)a^{k+1}}{2}\left(\frac{1}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + \frac{a^2}{k+2}\left(\frac{1}{a^{k+2}} - \frac{1}{(a+\frac{1}{\delta})^{k+2}}\right)\right.$$

$$\left. - \frac{2a}{k+1}\left(\frac{1}{a^{k+1}} - \frac{1}{(a+\frac{1}{\delta})^{k+1}}\right)\right)$$

$$= t_o + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+1}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{k+3}{2}\, a^{k+1}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) + \frac{a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)$$

$$+ \frac{k+1}{2}\, a^{k+3}\left(\frac{1}{a^{k+2}} - \frac{1}{(a+\frac{1}{\delta})^{k+2}}\right) - (k+2)a^{k+2}\left(\frac{1}{a^{k+1}} - \frac{1}{(a+\frac{1}{\delta})^{k+1}}\right)$$

$$= t_o + \frac{3a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right) - \frac{a^{k+1}}{(a+\frac{1}{\delta})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$+ \frac{ka^{k+1}}{a}\left(\frac{1}{k} - \frac{1}{\left(a+\frac{1}{\delta}\right)^{k}}\right) + \frac{3a^{k+1}}{2}\left(\frac{1}{a^{k}} - \frac{1}{\left(a+\frac{1}{\delta}\right)^{k}}\right)$$

$$+ \frac{(k+1)a^{k+3}}{2}\left(\frac{1}{a^{k+2}} - \frac{1}{\left(a+\frac{1}{\delta}\right)^{k+2}}\right) - (k+2)a^{k+2}\left(\frac{1}{a^{k+1}} - \frac{1}{\left(a+\frac{1}{\delta}\right)^{k+1}}\right)$$

$$+ t_o + \frac{3a^{k+1}}{k}\left(\frac{1}{a^{k}} - \frac{1}{\left(a+\frac{1}{\delta}\right)^{k}}\right) - \frac{(k+5)\,a^{k+1}}{2\left(a+\frac{1}{\delta}\right)^{k}}$$

$$+ \frac{(k+3)a^{k+2}}{\left(a+\frac{1}{\delta}\right)^{k+1}} - \frac{(k+1)\,a^{k+3}}{2\left(a+\frac{1}{\delta}\right)^{k+2}}$$

$$= f_3(a, k, \delta), \text{ say.} \qquad \ldots\ldots(4.22)$$

Numerical Illustration.

Using the estimates of a,k and δ as obtained in Chapter 3 viz.

$\hat{a} = 0.40,$

$\hat{k} = 1.5$

$\hat{\delta} = 0.002$

the estimates of expected time from Marriage to first birth, first to second birth and second to third birth (assuming one to one correspondence between conception and birth) with the further adjustment for decreasing fertility level with age so the parity level of the mother are given as follows:

$$E_o = \frac{a}{k-1} = \frac{0.40}{1.5-1} = 0.80 \qquad \text{...(4.23)}$$

$$E_1 = \pi + (a)^{k+1}\left[\frac{1}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right]$$

$$= 1+((1012)(\frac{1}{1.5}(\frac{1}{(.40)^{1.5}} - \frac{1}{(.42)^{1.5}})) \qquad \text{..(4.24)}$$

$$= \pi + 0.2667 = 1.2667$$

$$E_2 = 2\pi + \frac{2a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)$$

$$- \frac{a^{k+1}}{(a+\frac{1}{-})^k} + \frac{a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$= 2\pi + 0.5334 = 2.5334 \qquad \text{...(4.25)}$$

$$E_3 = 3\pi + \frac{3a^{k+1}}{k}\left(\frac{1}{a^k} - \frac{1}{(a+\frac{1}{\delta})^k}\right)$$

$$- \frac{(k+5)a^{k+1}}{2(a+\frac{1}{\delta})^k} + \frac{(k-3)a^{k+2}}{(a+\frac{1}{\delta})^{k+1}}$$

$$- \frac{(k+1)a^{k+3}}{2(a+\frac{1}{\delta})^{k+2}}$$

$$= 3\pi + 0.8 = 3.8. \qquad \text{....(4.26)}$$

Conclusion

A further modification of the results when marital exposure T is considered finite is given by Srestha and Biswas (1985).

Here the density function of the waiting time distribution from marriage to first conception is given by

$$f(x) = k a^k \sum_{k'=0}^{\infty} \frac{1}{(a+x+k'T)^{k+1}} \qquad \ldots(4.27)$$

$$\Rightarrow \quad E(X) = \frac{a}{k-1}\left(1 - \left(\frac{a}{a+T}\right)^{k-1}\right). \qquad \ldots(4.28)$$

The monthly probability of conception under (4.27) is given by

$$P_{j,j+1} = \int_j^{j+1} f(x)dx$$

$$= k a^k \sum_{k'=0}^{\infty} \int_j^{j+1} \frac{dx}{(x+a+k'T)^{k+1}}$$

$$= a^k \sum_{k'=0}^{\infty} \left(\frac{1}{(a+j+k'T)^k} - \frac{1}{(a+j+1+k'T)^k}\right) \qquad \ldots(4.29)$$

In particular for j=0

$$P_{0,1} = 1-\left(\frac{a}{a+1}\right)^k + \left(\frac{a}{a+T}\right)^k - \left(\frac{a}{a+1+T}\right)^k. \qquad \ldots(4.30)$$

However, the result for advanced parity groups viz. (1-2) and (2-3) are still unavailable. These results if available could have really highlighted how much additional precision is again when the assumption of infinite marital exposure is waived. For the first order of conception the gain in precision is considered to be negligible. As such the exercise has not much motivation.

A MULTISTATE MARKOV CHAIN MODEL FOR EVALUATING A STERILIZATION POLICY

Sterilization Policy for the reduction of the birth rate in a population is often adopted on the basis of the number of surviving children. Generally, a sterilization policy is implemented after a couple has a desirable number of children on assuming a certain hypothetical survival rates operating in future. However, this method has certain degree of arbitrariness : viz, the long term effect of mortality in deciding the number of surviving children may not be properly accounted for. An improved technique of assessing the long term impact of such sterilization policies is therefore to consider the joint effect of fertility and mortality. The present exercise has been made on the basis of the same rationale. We have developed a Multistate Markov Chain model corresponding to varying fertility and mortality intensities at different levels (States) of surviving children of the couple. The asymptotic probabilities of having a fixed number of children have been worked out.

Development of the Model :

Notations :

(i) $P_n(t)$ = Probability of n number of children upto a time t, $(n=0,1,2,3,\ldots\ldots)$.

(ii) $\lambda_n \Delta + O(\Delta)$ = Probability of having (n+1) children in $(t+\Delta)$ given that n number of children are surviving at time t. (n=0,1,2,3,.......).

(iii) $\mu_n \Delta + O(\Delta)$ = Probability of having (n-1) children in $(t+\Delta)$ given that n number of children are surviving at time t, (n= 0,1,2,3,......).

(iv) M = A possible upper bound of the number of children to a couple.

(v) (M-K+1) = Number of surviving children at the time of sterilization based on certain policy. $1 \leq k \leq (M+1)$

Then we have the Kolmogorov equations :

$$P_0(t+\Delta) = P_0(t)(1-\lambda_0\Delta + o(\Delta)) + P_1(t)(\mu_1\Delta + o(\Delta)) \quad \cdots (5.1)$$

$$P_n(t+\Delta) = P_n(t)\left[1-(\lambda_n+\mu_n)\Delta + o(\Delta)\right] + P_{n-1}(t)\left[\lambda_{n-1}\Delta + o(\Delta)\right] + P_{n+1}(t)\left[\mu_{n+1}\Delta + o(\Delta)\right] \quad \cdots (5.2)$$

$$0 < n \leq M-K$$

and

$$P_{M-K+1}(t+\Delta) = P_{M-K}(t)(\lambda_{M-K}\Delta + o(\Delta))$$
$$+ P_{M-K+1}(t)(1-\mu_{M-K+1}\Delta + o(\Delta)) \quad \ldots(5.3)$$

Now

$$(5.1) \Rightarrow P_0'(t) = -\lambda_0 P_0(t) + \mu_1 P_1(t) \quad \ldots(5.1')$$

$$(5.2) \Rightarrow P_n'(t) = -(\lambda_n+\mu_n)P_n(t) + P_{n-1}(t)\lambda_{n-1}$$
$$+ P_{n+1}(t)\mu_{n+1} \qquad 0<n\leq M-K \quad \ldots(5.2')$$

and

$$(5.3) \Rightarrow P_{M-K+1}'(t) = -\mu_{M-K+1} P_{M-K+1}(t) + \lambda_{M-K} P_{M-K}(t) \quad \ldots(5.3)'$$

Steady State Solution :

Steady state solutions of $(5.1')$, $(5.2')$ and $(5.3')$ are obtainable by assuming

$$P_0(\infty) = P_n(\infty) = P_{M-K+1}(\infty) = 0.$$

Applying these conditions and solving successively the $(M-K+2)$ equations, the parameters $\lambda_0, \lambda_1, \ldots\ldots, \lambda_{M-K+1}$, $\mu_1, \mu_2, \ldots\ldots, \mu_{M-K}$ parameters are obtained as follows :-

$$P_0(\infty) = \pi_0 = \left[1+\frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots + \frac{\lambda_0\lambda_1\lambda_2\cdots\lambda_{M-K}}{\mu_1\mu_2\mu_3\cdots\mu_{M-K+1}}\right]^{-1}$$

$$P_n(\infty) = \frac{\lambda_0 \lambda_1 \lambda_2 \cdots \lambda_{n-1}}{\mu_1 \mu_2 \cdots \mu_n} \pi_0 \quad ; \; 0 < n \leq M-K$$

$$P_{M-K+1}(\infty) = \Big[1 + \frac{\mu_{M-K+1}}{\lambda_{M-K}} + \frac{\mu_{M-K+1}\, \mu_{M-K}}{\lambda_{M-K}\, \lambda_{M-K+1}} + \cdots$$

$$\cdots + \frac{\mu_{M-K+1}\, \mu_{M-K} \cdots \mu_2 \mu_1}{\lambda_{M-K}\, \lambda_{M-K-1} \cdots \lambda_1 \lambda_0} \Big]^{-1}$$

<u>Examples</u> :

<u>Case I</u> : $M-K+1 = 2.$

i.e. sterilization is performed on attainment of two surviving children. Parameter matrix is $\begin{pmatrix} \lambda_0 & \lambda_1 \\ \mu_1 & \mu_2 \end{pmatrix}$

$$\pi_0 = \Big[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} \Big]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{\mu_1} \pi_0$$

$$\pi_2 = \Big[1 + \frac{\mu_2}{\lambda_1} + \frac{\mu_2 \mu_1}{\lambda_1 \lambda_0} \Big]^{-1}$$

<u>Case II</u> : $M-K+1 = 3.$

i.e. sterilization is performed on attainment of three surviving children. Parameter matrix is $\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 \\ \mu_1 & \mu_2 & \mu_3 \end{pmatrix}$

$$\pi_0 = \left[1 + \frac{\lambda_0}{M_1} + \frac{\lambda_0 \lambda_1}{M_1 M_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{M_1 M_2 M_3}\right]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{M_1} \pi_0$$

$$\pi_2 = \frac{\lambda_0 \lambda_1}{M_1 M_2} \pi_0$$

$$\pi_3 = \left[1 + \frac{M_3}{\lambda_2} + \frac{M_3 M_2}{\lambda_2 \lambda_1} + \frac{M_3 M_2 M_1}{\lambda_2 \lambda_1 \lambda_0}\right]^{-1}$$

<u>Case III</u> : M-K+1 = 4.

i.e. sterilization is made on attainment of four surviving children. Parameter matrix is $\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 \\ M_1 & M_2 & M_3 & M_4 \end{pmatrix}$

$$\pi_0 = \left[1 + \frac{\lambda_0}{M_1} + \frac{\lambda_0 \lambda_1}{M_1 M_2} + \frac{\lambda_0 \lambda_1 \lambda_2}{M_1 M_2 M_3} + \frac{\lambda_0 \lambda_1 \lambda_2 \lambda_3}{M_1 M_2 M_3 M_4}\right]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{M_1} \pi_0$$

$$\pi_2 = \frac{\lambda_0 \lambda_1}{M_1 M_2} \pi_0$$

$$\pi_3 = \frac{\lambda_0 \lambda_1 \lambda_2}{M_1 M_2 M_3} \pi_0$$

$$\pi_4 = \left[1 + \frac{M_4}{\lambda_3} + \frac{M_4 M_3}{\lambda_3 \lambda_2} + \frac{M_4 M_3 M_2}{\lambda_3 \lambda_2 \lambda_1} + \frac{M_4 M_3 M_2 M_1}{\lambda_3 \lambda_2 \lambda_1 \lambda_0}\right]^{-1}$$

<u>Case IV</u> : M-K+1 = 5.

i.e. sterilization is performed on attainment of five surving children. Parameter matrix is $\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ M_1 & M_2 & M_3 & M_4 & M_5 \end{pmatrix}$

$$\pi_0 = \left[1 + \frac{\lambda_0}{M_1} + \frac{\lambda_0\lambda_1}{M_1M_2} + \frac{\lambda_0\lambda_1\lambda_2}{M_1M_2M_3} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{M_1M_2M_3M_4} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{M_1M_2M_3M_4M_5}\right]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{M_1}\pi_0$$

$$\pi_2 = \frac{\lambda_0\lambda_1}{M_1M_2}\pi_0$$

$$\pi_3 = \frac{\lambda_0\lambda_1\lambda_2}{M_1M_2M_3}\pi_0$$

$$\pi_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{M_1M_2M_3M_4}\pi_0$$

$$\pi_5 = \left[1 + \frac{M_5}{\lambda_4} + \frac{M_5M_4}{\lambda_4\lambda_3} + \frac{M_5M_4M_3}{\lambda_4\lambda_3\lambda_2} + \frac{M_5M_4M_3M_2}{\lambda_4\lambda_3\lambda_2\lambda_1} + \frac{M_5M_4M_3M_2M_1}{\lambda_4\lambda_3\lambda_2\lambda_1\lambda_0}\right]^{-1}$$

Case V: M-K+1 = 6.

i.e. sterilization is performed on attainment of six surviving children. Parameter matrix is $\begin{pmatrix} \lambda_0 & \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 \\ M_1 & M_2 & M_3 & M_4 & M_5 & M_6 \end{pmatrix}$

$$\pi_0 = \left[1 + \frac{\lambda_0}{M_1} + \frac{\lambda_0\lambda_1}{M_1M_2} + \frac{\lambda_0\lambda_1\lambda_2}{M_1M_2M_3} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{M_1M_2M_3M_4} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{M_1M_2M_3M_4M_5} + \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4\lambda_5}{M_1M_2M_3M_4M_5M_6}\right]^{-1}$$

$$\pi_1 = \frac{\lambda_0}{M_1}\pi_0$$

$$\pi_2 = \frac{\lambda_0\lambda_1}{M_1M_2}\pi_0$$

$$\pi_3 = \frac{\lambda_0\lambda_1\lambda_2}{M_1M_2M_3}\pi_0$$

$$\pi_4 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3}{M_1M_2M_3M_4}\pi_0$$

$$\pi_5 = \frac{\lambda_0\lambda_1\lambda_2\lambda_3\lambda_4}{M_1M_2M_3M_4M_5}\pi_0$$

$$\pi_6 = \left[1 + \frac{M_6}{\lambda_5} + \frac{M_6 M_5}{\lambda_5 \lambda_4} + \frac{M_6 M_5 M_4}{\lambda_5 \lambda_4 \lambda_3} + \frac{M_6 M_5 M_4 M_3}{\lambda_5 \lambda_4 \lambda_3 \lambda_2} + \frac{M_6 M_5 M_4 M_3 M_2}{\lambda_5 \lambda_4 \lambda_3 \lambda_2 \lambda_1} + \frac{M_6 M_5 M_4 M_3 M_2 M_1}{\lambda_5 \lambda_4 \lambda_3 \lambda_2 \lambda_1 \lambda_0} \right]^{-1}$$

For presenting the effect of different levels of sterilization at various parity levels of the couples (from second to sixth surviving children), we define the fertility and mortality classes as follows :

Fertility and Mortality (F.M.) Class I :

$\lambda_0 = 0.20$, $\lambda_1 = 0.15$, $\lambda_2 = 0.12$, $\lambda_3 = 0.10$,
$\lambda_4 = 0.08$, $\lambda_5 = 0.06$;

$M_1 = 0.05$, $M_2 = M_3 = M_4 = M_5 = M_6 = 0.04$.

FM Class II :

$\lambda_0 =$ 0.30, $\lambda_1 = 0.25$, $\lambda_2 = 0.22$, $\lambda_3 = 0.20$
$\lambda_4 = 0.18$, $\lambda_5 = 0.16$;

$M_1 =$ 0.06, $M_2 = M_3 = M_4 = M_5 = M_6 = 0.05$;

<u>FM Class III</u> :

$$\lambda_o = 0.40, \quad \lambda_1 = 0.35, \quad \lambda_2 = 0.32, \quad \lambda_3 = 0.30,$$

$$\lambda_4 = 0.28, \quad \lambda_5 = 0.26;$$

$$\mu_1 = 0.07, \quad \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0.06;$$

<u>FM Class IV</u> :

$$\lambda_o = 0.50, \lambda_1 = 0.45, \lambda_2 = 0.42, \quad \lambda_3 = 0.40$$

$$\lambda_4 = 0.38, \quad \lambda_5 = 0.36;$$

$$\mu_1 = 0.08, \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0.07;$$

<u>FM Class V</u> :

$$\lambda_o = 0.60, \lambda_1 = 0.55, \quad \lambda_2 = 0.52, \lambda_3 = 0.50,$$

$$\lambda_4 = 0.48, \quad \lambda_5 = 0.46;$$

$$\mu_1 = 0.09 \, \mu_2 = \mu_3 = \mu_4 = \mu_5 = \mu_6 = 0.08;$$

Steady state probability distributions of the number of surviving children classified by different levels of sterlization are given below :

Table 5.1 (F.M. Class I)

Number of surviving children	Level of 2	sterilization at parties 3	4	5	6
0	0.0500	0.0154	0.0056	0.0025	0.0014
1	0.2000	0.0615	0.0225	0.0099	0.0054
2	0.7500	0.2308	0.0845	0.0373	0.0203
3	_	0.6923	0.2535	0.1118	0.0608
4	_	_	0.6338	0.2795	0.1520
5	_	_	_	0.5590	0.3041
6	_	_	_	_	0.4561

Table 5.2 (F.M. Class II)

Number of surviving children	Level of sterilization at parities 2	3	4	5	6
0	0.0323	0.0071	0.0017	0.0005	0.0001
1	0.1613	0.0355	0.0086	0.0023	0.0007
2	0.8065	0.1773	0.0430	0.0115	0.0035
3	-	0.7801	0.1893	0.0508	0.0152
4	-	-	0.7573	0.2032	0.0608
5	-	-	-	0.7316	0.2190
6	-	-	-	-	0.7007

Table 5.3 (F.M Class III)

Number of surviving children	Level of sterilization at parities 2	3	4	5	6
0	0.0250	0.0046	0.0009	0.0002	0.0000
1	0.1427	0.0262	0.0052	0.0011	0.0002
2	0.8323	0.1530	0.0301	0.0063	0.0014
3	-	0.8161	0.1606	0.0338	0.0077
4	-	-	0.8032	0.1692	0.0383
5	-	-	-	0.7894	0.1786
6	-	-	-	-	0.7738

Table 5.4 (F.M. Class IV)

Number of surviving children	Level of sterilization at parities 2	3	4	5	6
0	0.0211	0.0035	0.0006	0.0001	0.0000
1	0.1318	0.0217	0.0038	0.0007	0.0001
2	0.8471	0.1393	0.0241	0.0044	0.0008
3	-	0.8256	0.1447	0.0264	0.0051
4	-	-	0.8268	0.1506	0.0289
5	-	-	-	0.8178	0.1571
6	-	-	-	-	0.8079

Table 5.5 (F.M. Class V)

Number of surviving children	Level of sterilization at parities 2	3	4	5	6
0	0.0187	0.0028	0.0005	0.0001	0.0000
1	0.1246	0.0190	0.0030	0.0005	0.0001
2	0.8567	0.1304	0.0207	0.0034	0.0006
3	-	0.8478	0.1346	0.0223	0.0038
4	-	-	0.8412	0.1391	0.0240
5	-	-	-	0.8346	0.1439
6	-	-	-	-	0.8276

Discussion :

An overall glance in the table 5.1 to 5.5 reveals that the proportion of surviving children at all levels of fertility and mortality classes under consideration decreases consistently when sterilization levels are relaxed. For example, Table I (F.M. Class I) shows that the proportion of couples having two surviving children in the long run is 75% when sterilization is made on attainment of two surviving children as against only 45% of the couples having six surviving children when sterilization level is relaxed to the extent of sterilizing couples only after having six surviving children.

The same trend is more or less revealed in all other tables, corresponding to different sets of fertility and mortality levels as may immediately be seen on inspection of the diagonal elements of the matrices corresponding to each of the tables from I to V. However, a better insight into the variation of the proportion of surviving children as a result of changes in the level of parity under consideration for adopting a sterilization policy may be obtained when we consider the average no of surviving children under different F.M. set up. This is considered in the following table.

Table 5.6

Mean number of surviving children at different levels of sterilization

F.M Class	Level of sterilization at number of surving children 2	3	4	5	6
I	1.7000	2.6000	3.4872	4.3329	5.0935
II	1.7743	2.7304	3.6917	4.6485	5.5957
III	1.8073	2.7805	3.7600	4.7389	5.7151
IV	1.8260	2.8071	3.7933	4.7801	5.7655
V	1.8380	2.8232	3.8130	4.8036	5.7935

Table 5.6 provides a clue for deciding the optimal level of sterilization depending on the number of surviving

children. For example, if the policy is to ensure at least one surving male child in the long run (as it is most customary in India) per couple then it may be seen that irrespective of the variations in the levels of the fertility and mortality the level of sterilization may optimally be made on the basis of three surviving children. Whereas if sterilization policy is relaxed to the extent of sterilizing couples only on the basis of more than three number of surviving children then one may expect a phenomenal increase in the growth rate of the population not consistent with the situation leading to adopt a sterilization policy motivated to cut down the growth rate of the population significantly. On the other hand if sterilization policy is made on the basis of only two surviving children then the condition of ensuring at least one male child in the long run per couple may not be realized although the same may fulfil the plan of reducing the growth rate in a fast increasing population. Such a policy may not be culturally and traditionally accepted by the people at large.

CHAPTER-6

ON A MARTINGALE APPROACH TO A PROBLEM ON STERILIZATION POLICY

6.1 Introduction

Suppose we have a problem of sterilization of mothers with i_0 number of surviving children, attained by addition of a birth at the starting point of the observation. These mothers may be considered homogeneous with respect to ages, i_0 is a minimum number which makes the mother eligible for sterilization.

However, if i_0 is reduced to $(i_0 -1)$ first, then the mother concerned, is discarded for sterilization once for all. Whereas those mothers whose number of surviving children move from i_0 to $(i_0 +1)$ first (without passing through any other state) is considered eligible for sterilization and they are sterilized at a random time S_T on attainment of $(i_0 +1)$ numbers of surviving children. Besides these two groups, the proportion of women who will remain in state S_i number of surviving children throughout a period T from the starting point of the observation (the reference period π which is considered large enough to cover all the three types of outcomes) will be discarded from the sterilization programme.

The asymptotic estimator (for large T) of the above have been attempted to be obtained using Martingale stopping rule.

6.2 Development of the Model

Notations

(i) $X(t)$ = Number of surviving children at any time t of the birth and death process with birth and death parameters :

$\lambda_i = \lambda(i)$ and $\mu_i = \mu(i)$ respectively,

i being the size of the population.

(ii) $\pi_0 = P[X(t) = i_0 - 1 \mid X(0) = i_0 ;\ X(j) = i_0$ for all $j < t]$. π_0 being the asymtotic probability

(iii) $\alpha(T) = P[X(j) = i_0 \mid X(0) = i_0$ for all $j \in (0,T)]$.

(iv) $P_k(t) = P[X(t) = k]$; $k = 0,1,2,\ldots\ldots$

We have

$$P_k(t+\Delta) = P_k(t)(1-(\lambda_k \Delta + \mu_k \Delta) + P_{k-1}(t)\lambda_{k-1}\Delta + P_{k+1}(t).$$

$$.\mu_{k+1}\Delta + O(\Delta)) \qquad \ldots\ldots(6.1)$$

for $k = 0,1,2,\ldots\ldots,$

subject to $P_{k-1}(t) = 0$; for $k=0$.

Let us define

$$f(j) = 1 + \frac{\mu_1}{\lambda_1} + \frac{\mu_1\,\mu_2}{\lambda_1\,\lambda_2} + \ldots + \frac{\mu_1\,\mu_2\ldots\mu_{j-1}}{\lambda_1\,\lambda_2\ldots\lambda_{j-1}} \qquad \ldots(6.2)$$

Corresponding to the parameters $\lambda_i = \lambda(i)$ and $\mu_i = \mu(i)$, $i = 1,2,3,\ldots\ldots$

Then it can immediately be shown that $f(X(t))$ is a Martingale with respect to $\mathcal{F}_t = \mathcal{F}(X(t)$ (a σ-field) $\mathcal{F}_u = \mathcal{F}(X(u))$; $0 \leq u \leq t$ (Karlin and Taylor (1975)). Suppose we define a stopping time S_T

$$S_T - \min.(t \geq 0.X(t) = i_0-1; X(t) = i_0+1\ X(0) = i_0)$$

for a given T where $T=S_T$ ....(6.3)

S_T represents the stopping time when $X(t) = i_0 - 1$, or $X(t) = i_0 + 1$ or $X(t) = i_0$ for all $(0,T]$. In this case $S_T = T$, and constant time T is a Markov time which are considered as absorbing states of the process given $X(0) = i_0$. Then by optional sampling theorem of Martingale (Karlin and Taylor (1975)).

$$E(f(x(S_T))) = E(f(XO)) = E(f(i_0)) = f(i_0). \quad ....(6.4)$$

since $X(0) = i_0$, a fixed number by assumption.

Then

$$E(f(i_0)) = f(i_0) = E(f(X_T))$$

$$= (1-\pi_0)\left[\alpha f(i_0) + (1-\alpha) f(i_0+1)\right] + \pi_0 (f(i_0-1)) \quad ......(6.5)$$

$$\Rightarrow \pi_0 = \frac{f(i_0) - (1-\alpha) f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha) f(i_0+1) - \alpha f(i_0)} \quad .....(6.6)$$

Then π_0 proportion will not be sterilized and $(1-\alpha)$ $(1-\pi_0)$ will be sterilized where

$$\alpha = e^{-(\lambda_{i_0} - M_{i_0})T} \quad \text{for large } T \qquad \cdots (6.7)$$

the reference period in the sterilization programme.

Particular Case

Let $i_0 = 3$;

$$f(i_0 - 1) = 1 + \frac{M_1}{\lambda_1}$$

$$f(i_0) = 1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2}$$

$$f(i_0 + 1) = 1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2} + \frac{M_1 M_2 M_3}{\lambda_1 \lambda_2 \lambda_3}$$

$$\pi_0 = \frac{\left[1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2}\right] - \left[\left(1 - e^{-(\lambda_3 - M_3)T}\right)\right]\left[1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2} + \frac{M_1 M_2 M_3}{\lambda_1 \lambda_2 \lambda_3}\right] - e^{-(\lambda_3 - M_3)T}\left[1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2}\right]}{\left\{\left[1 + \frac{M_1}{\lambda_1}\right] - \left(1 - e^{-(\lambda_3 - M_3)T}\right)\left(1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2} + \frac{M_1 M_2 M_3}{\lambda_1 \lambda_2 \lambda_3}\right) - e^{-(\lambda_3 - M_3)T}\left(1 + \frac{M_1}{\lambda_1} + \frac{M_1 M_2}{\lambda_1 \lambda_2}\right)\right\}} \qquad \cdots (6.8)$$

Therefore, the total proportion of mothers not to be sterilized

$$= \left[\alpha(1 - \pi_0) + \pi_0\right]$$

where $\alpha = e^{-(\lambda_{i_0} - M_{i_0})T}$ for given large T ..(6.9)

Next, to obtain the expected stopping time $S_{T'}$ which decides the expected time when policy decision is taken for sterilization, 100 $(1-\alpha)$ $(1-\pi_0)$ % of mothers while leaving $100[\alpha(1-\pi_0) + \pi_0]$% mothers for sterilization programme. We have,

$$Y(t) = g(X(t))-\int_0^t \lambda(X(u))(g(X(u)+1)-g(X|u))$$

$$- \mu(x(u)) \quad (g(X(u))-g(X(u)-1)))du, \qquad \ldots(6.10)$$

is a Martingale over $\mathcal{F}_t = \mathcal{F}(X(t))$ for arbitrary function g, provided the expectation of $Y(t)$ exists. Further, if g(i) be so chosen, so that

$$\lambda(i)[g(i+1)-g(i)]-\mu(i)[g(i)-g(i-1)] = 1 \qquad \ldots.(6.11)$$

for all i = 1,2,3......

then

$$Y(t) = (g(X(t))-t) \text{ is a Martingale}$$

$$\Rightarrow \quad E(Y(S_T)) = E(Y(O)) = E[g(X(t))]-E(S_T) \qquad \ldots.(6.12)$$

by Martingale optional sampling theorem.

Further, $Y(O) = g(X(O)) = g(i_o)$

$$= \quad g(i_o) = E(g(X(S_T))) - E(S_T) \qquad \ldots..(6.13)$$

$$= \quad E(g(X(S_T)))-g(i_o) = E(S_T). \qquad \ldots.(6.14)$$

To obtain $E(S_T)$, we require the solution of the difference equation given by (6.11).

Putting, $g(i+1)-g(i) = \Delta g(i) = v(i)$.

We have the transformed equation of (6.11) as

$$(\lambda(i)\, V(i) - \mu(i)\, V(i)) = 1. \qquad \text{....(6.11')}$$

We may immediately get

$$V(0) = 0, V(1) = (g(2)-g(i)) = \frac{1}{\lambda_1}$$

$$V(2) = (g(3)-g(2)) = \frac{1}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right)$$

$$V(3) = (g(4)-g(3)) = \frac{1}{\lambda_3}\left[1+\frac{\mu_3}{\lambda_2}\left(1+\frac{1}{\lambda_1}\right)\right]$$

and in general

$$V(i+1) = \frac{1}{\lambda_i}\left[1+\frac{\mu_i}{\lambda_{i-1}}\left(1+\frac{\mu_{i-1}}{\lambda_{i-2}}\left(1+\frac{\mu_{i-2}}{\lambda_{i-3}}\left(1+\frac{\mu_{i-3}}{\lambda_{i-4}}\left(1+\frac{\mu_{i-4}}{\lambda_{i-5}}\right(\ldots\right.\right.\right.\right.\right.$$

$$\ldots\ldots)\left(1+\frac{\mu_2}{\lambda_1}\right); \quad \text{for } i \geq 1$$

$$\text{.....(6.12')}$$

This gives the solution $g(i)$ of (6.11) as

$$g(i+1) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right) + \frac{1}{\lambda_3}\left(1+\frac{\mu_3}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right)\right) + \cdots$$

$$+ \frac{1}{\lambda_i}\left[1+\frac{\mu_i}{\lambda_{i-1}}\left(1+\frac{\mu_{i-2}}{\lambda_{i-3}}\left(1+\frac{\mu_{i-4}}{\lambda_{i-5}}\right)+\cdots\right.\right.$$

$$\left.\left(1+\frac{\mu_3}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right)\right)\right] \quad \text{for } i \geq 1 \qquad \text{...(6.13')}$$

and g(1) = 0, g(0) = 0 etc.

In particular, for $i_0 = 3$, $g(i_0 - 1) = \frac{1}{\lambda_1}$

$$g(i_0) = g(3) = \left(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}\left(1 + \frac{\mu_2}{\lambda_1}\right)\right)$$

$$g(i_0+1) = g(4) = \frac{1}{\lambda_1} + \frac{1}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right) + \frac{1}{\lambda_3}\left(1+\frac{\mu_3}{\lambda_2}\left(1+\frac{\mu_2}{\lambda_1}\right)\right)$$

$E(g(X(S_T))) =$

$$(1-\pi_0)(1-\alpha)g(i_0+1) + \alpha(1-\pi_0)g(i_0) + \pi_0 g(i_0+1)$$

$$= \left(1 - \left[\frac{f(i_0) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}\right](1-\alpha)g(i_0+1)\right)$$

$$+ \alpha\left(1 - \left[\frac{f(i_0) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}\right] g(i_0)\right)$$

$$+ \left[\frac{f(i_0) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}\right] g(i_0-1)$$

$$= \frac{f(i_0-1) - f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}(1-\alpha)g(i_0+1)$$

$$+ \frac{f(i_0-1) - f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}\alpha g(i_0)$$

$$+ \frac{f(i_0) - (1-\alpha)f(i_0+1) - \alpha f(i_0)}{f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0)} g(i_0-1)$$

$$= \left[\frac{((f(i_0-1)-f(i_0))(1-\alpha))g(i_0-1) + (\alpha f(i_0-1) - f(i_0))g(i_0) + \left[f(i_0)(1-\alpha)-(1-\alpha)f(i_0+1)\right]g(i_0-1)}{(f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0))} \right]$$

For, $i_0 = 3$

$$E(S_T) = E(g(X_{S_T})) - g(i_0)$$

$$= \left[\frac{\left[(f(i_0-1)-f(i_0))(1-\alpha)\right]g(i_0+1) + \left[\alpha(f(i_0-1)-f(i_0))\right] \cdot g(i_0) + \left[(1-\alpha)f(i_0)-(1-\alpha)f(i_0+1)\right]g(i_0-1)}{(f(i_0-1) - (1-\alpha)f(i_0+1) - \alpha f(i_0))} - g(i_0) \right]$$

where f, g and α are given in (6.2), (6.12) and (6.7) respectively.

6.3 A Numerical Illustration

Let

$\lambda_1 = .20$, $\mu_1 = .05$ and $T = 3$ years,

$\lambda_2 = .16$, $\mu_2 = .06$

$\lambda_3 = .14$, $\mu_3 = .07$

$\lambda_4 = .10$, $\mu_4 = .08$.

The values of π_0, α, and $E(S_T)$ have been tabulated in table I, for i=2,3,4 to consider the suitability of the initial number i_0 to be chosen for the sterilization programme.

Table 6.1

i_0	π_0	α	$E(S_T)$	$(i-\pi_0)(1-\alpha)$
(1)	(2)	(3)	(4)	(5)
2	0.0885	0.7408	1.476486	0.2363
3	0.0868	0.8106	1.121893	0.17296
4	0.0448	0.9418	0.154163	0.0556

The values of π_0, α and $E(S_T)$ for different values of i, under hypothetical values of $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ and $(\mu_1, \mu_2, \mu_3, \mu_4)$.

6.4 Conclusion

A glance in the Table 6.1 reveals the fact for i_0 = 2, may be the suitablemost starting number of surviving children in this kind of sterilization programme under the hypothetical values of the parameters. This corresponds to about 24% of the mothers with 2 living children who may be sterilized. However, it will be interesting to employ

live data providing estimates of λ_i and μ_i, (i = 2,3,4) and this decide the policy of the sterilization programme.

CHAPTER-7

ON THE EVOLUTION OF DIRECT AND INDIRECT STRATEGIES FOR THE REDUCTION OF CONSCEPTION RATE - A COUNTER MODEL APPROACH

7.0. The efficacy of any family limitation strategy has to be judged primarily on its effect on the interval between two consecutive order of conceptions. Given that a conception has occured at the beginning of the interval, following which there exists an infecundable period (dead time in the parlance of Geiger Muller Counter Model theory); which is partially composed of a fixed gestation period (in case of a full term live birth) as well as a random period corresponding to the Post Partum Amenorrhoea (P.P.A). The period of Post Partum Amenorrhoea (P.P.A) is followed by the resumption of fecundable period which continues till a subsequent conception takes place. Hence statistically the interval between two consecutive order of conceptions can be conceived as the convolution of two probability distribution; the first one corresponds to the infecundable period (a part of which is again fixed) and the other to the fecundable period during which a woman is exposed to the risk of conceptions with the completion of the gestation period point chosen as origin.

7.1. Development of the Model :

Assuming that the Gestation period is fixed and is of length π(= 9 months for a live birth) and the period of P.P.A. conforms to a negative exponential distribution with parameter λ (Talwar (1964)) and the waiting time distribution for a subsequent conception following the resumption of the fecundable period is also a negative exponential with parameter (Singh (1964)) the waiting time distribution becomes a convolution of two negative exponential with parameter λ and μ respectively. The convoluted density function is given by

$$\psi(t) = \int_0^t g(y)\, f(t-y)dy$$

where

$$g(y) = \lambda e^{-\lambda y}; \quad 0<\lambda<\infty$$
$$f(t-y) = \mu e^{-\mu(t-y)}; \quad 0<\mu<\infty.$$

Denoting the convolution of g and f as $g * f$

$$g * f = \lambda \int_0^t e^{-\lambda y} \mu e^{-\mu(t-y)} dy$$
$$= \lambda\mu e^{-\mu t} \int_0^t e^{-(\lambda-\mu)y} dy$$

$$= \lambda \mu e^{-\mu t} \left[\frac{e^{-(\lambda-\mu)y}}{-(\lambda-\mu)} \right]_0^t$$

$$= \lambda \mu\, e^{-\mu t} \left[\frac{e^{-(\lambda-\mu)t}}{-(\lambda-\mu)} + \frac{1}{\lambda-\mu} \right]$$

$$= \frac{\lambda \mu}{\lambda-\mu} \left(e^{-\mu t} - e^{-\lambda t} \right)$$

....(7.1)

Again shifting the origin to π to take account of the gestation prior to the period of Post Partum Amenorrhoea (which implies that the range of the random variable of the infecundable period plus the fecundable period t lies between π to ∞ i.e. $\pi \leq t < \infty$), we have the expected time between two consecutive order of conceptions as

$$E(T) = \pi + \frac{\left(\frac{\lambda \mu}{\lambda-\mu}\right) \int_{\pi}^{\infty} t \left[e^{-\mu t} - e^{-\lambda t} \right] dt}{\left(\frac{\lambda \mu}{\lambda-\mu}\right) \int_{\pi}^{\infty} \left[e^{-\mu t} - e^{-\lambda t} \right] dt}$$

......(7.2)

$$E(T) = \pi + \frac{\frac{\pi}{\mu}(e^{-\mu\pi}) - \frac{\pi}{\mu}(e^{-\lambda\pi}) + \frac{1}{\mu^2}e^{-\mu\pi} - \frac{1}{\lambda^2}e^{-\lambda\pi}}{\frac{e^{-\mu\pi}}{\mu} - \frac{e^{-\lambda\pi}}{\lambda}} \qquad \ldots.(7.3)$$

Further assuming that the parameters λ and μ vary , over different order of conceptions and denoting λ_i and μ_i as the corresponding Poisson intensities (rate) for going from Infecundable period to Fecundable period and going from Fecundable period to Infecundable period respectively between $(i-1)^{th}$ and i^{th} order of conceptions, $i = 1,2,3$....... We have

$$\nu(c_{i-1}, c_i) = \pi + \frac{\frac{\pi}{\mu_i}(e^{-\mu_i\pi}) - \frac{\pi}{\lambda_i}(e^{-\lambda_i\pi}) + \frac{1}{\mu_i^2}e^{-\mu_i\pi} - \frac{1}{\lambda_i^2}e^{-\lambda_i\pi}}{\frac{e^{-\mu_i\pi}}{\mu_i} - \frac{e^{-\lambda_i\pi}}{\lambda_i}} \qquad \ldots(7.4)$$

where $\nu(c_{i-1}, c_i)$ represents the interval between the $(i-1)^{th}$ and i^{th} order of conceptions.

Methods for increasing the mean conception interval $\nu(c_{i-1}, c_i)$ $i = 1,2,3,\ldots$ can be realised in two different ways: (A) By reducing the intensity of conception μ_i $(i=1,2,3\ldots.)$ (direct Family planning (F.P) methods) (B) By reducing the intensity λ_i of going from infecundable period to fecundable period $(i=1,2,3\ldots.)$. (Indirect method of Family Planning can be realised by increasing the period of lactation etc.).

<u>under (A)</u> if the conception rate μ_i' is reduced to $\mu_i e^{-\delta_i}$ $\delta_i > 0$ i = 1,2,3,.... where $e^{-\delta_i}$ is considered to be a damping factor which may vary from one order of conception to other depending on Socio-Economic and cultural factors. The increased interval under (A) is given by

$$\nu'(c_{i-1}, c_i) = \pi + \frac{\dfrac{\pi}{\mu_i e^{-\delta_i}}\left(e^{-\mu_i e^{-\delta_i}\pi}\right) - \dfrac{\pi}{\lambda_i}\left(e^{-\lambda_i \pi}\right) + \dfrac{e^{-\mu_i \pi e^{-\delta_i}}}{\mu_i^2 e^{-2\delta_i}} - \dfrac{e^{-\lambda_i \pi}}{\lambda_i^2}}{\dfrac{e^{-\mu_0 e^{-\delta_i}\pi}}{\mu_i e^{-\delta_i}} - \dfrac{e^{-\lambda_i \pi}}{\lambda_i}} \quad \ldots (7.5)$$

<u>under (B)</u> if the intensity λ_i is reduced to $\lambda_i e^{-\Delta_i}$ $\lambda_i > 0$ and i = 1,2,.......

then the increased interval under (B) is given by

$$\nu''(c_{i-1}, c_i) = \pi + \frac{\frac{\pi}{\mu_i}\left(e^{-\mu_i\pi}\right) - \frac{\pi}{\lambda_i e^{-\Delta_i}} e^{-\lambda_i e^{-\Delta_i\pi}} + \frac{e^{-\mu_i\pi}}{\mu_i^2} - \frac{e^{-\lambda_i e^{-\Delta_i\pi}}}{\lambda_i^2 e^{-2\Delta_i}}}{\frac{e^{-\mu_i\pi}}{\mu_i} - \frac{e^{-\lambda_i e^{-\Delta_i\pi}}}{\lambda_i e^{-\Delta_i\pi}}} \quad \ldots(7.6)$$

Finally under (A) and (B) implemented together the increased interval between (i-1)th and ith order of conceptions is given by

$$\nu'''(c_{i-1}, c_i) = \pi + \frac{\frac{\pi}{\mu_i e^{-\delta_i}} e^{-\mu_i e^{-\delta_i\pi}} - \frac{\pi e^{-\lambda_i e^{-\Delta_i\pi}}}{\lambda_i e^{-\Delta_i}} + \frac{e^{-\mu_i e^{-\delta_i\pi}}}{\mu_i^2 e^{-2\delta_i}} - \frac{e^{-\lambda_i e^{-\Delta_i\pi}}}{\lambda_i^2 e^{-2\Delta_i}}}{\left(e^{-\mu_i e^{-\delta_i\pi}}/\mu_i e^{-\delta_i}\right) - \left(e^{-\lambda_i e^{-\Delta_i\pi}}/\lambda_i e^{-\Delta_i}\right)} \quad \ldots(7.7)$$

Formulation of (7.7) is based on the rationale of absence of any interaction between direct and indirect methods of family limiation.

7.2. Numerical Exercise on the Basis of Data on South Indian Women :

The increase in the interval for order of conceptions (1-2) (2-3) and (3-4) under (A) and (B) are examined in the light of the data based on a sample of women in South India (Srinivasan (1972)) details of which have been quoted by Mitra (1982) as follows

Ta^le 7 1

Monthly Probability of Conceptions and Other Parameters of Indian Womens

Parameters	Interval between conceptions* First & Second	Second & Third	Third and Fourth
1	2	3	4
Sample size	297	306	245
Average interval (in months)	36.1	38.9	38.6
Variance of the interval(in months)	345.8	407.8	380.1
Monthly probability of conception( )	.052	.048	.050
Non Susceptible Period (in months)	17.0	18.2	18.6

The estimates of $\hat{\mu}_i$ (i = 1,2,3) are taken from the figure given in the third row of the table 5.1.

* This is actually the interval between two consecutive order of births which is the same as the waiting time between two consecutive order of conceptions under the assumption of one to one correspondence between conception and live births.

The estimates of $\hat{\lambda}_i$'s $(i = 1,2,3)$ are obtainable by deducting the constant infecundable period due to gestation as 9 months and taking the reciprocal of the residual.

Further the assumptions relating to (A) and (B) were made as follows :

$$e^{-\delta} = 0.90,\ 0.85,\ 0.80,\ 0.75 \text{ and } 0.70$$

$$e^{-\Delta} = 0.90 \text{ and } 0.95$$

$$\pi = 9.$$

Table 7.3 A

showing change in Inter birth Interval (1-2) as a result of different levels intervention by direct and indirect methods

BIRTH INTERVAL IN MONTHS BETWEEN 1ST AND 2ND BIRTHS

Degree of Intervention Indirect	Direct	Without intervention	With indirect intervention	With direct interv-ention	with direct & indirect intervention
.95	.90	40.32	40.65	42.41	42.73
.95	.85	40.32	40.65	43.64	43.96
.95	.80	40.32	40.65	45.03	45.35
.95	.75	40.32	40.65	46.61	46.93
.95	.70	40.32	40.65	48.41	48.73
.90	.90	40.32	41.02	42.41	43.10
.90	.85	40.32	41.02	43.64	44.33
.90	.80	40.32	41.02	45.03	45.72
.90	.75	40.32	41.02	46.61	47.29
.90	.70	40.32	41.02	48.41	49.09

Table 7.3 B

showing the change in the Inter birth interval (2-3) as a result of different levels of intervention by direct and indirect methods

BIRTH INTERVAL IN MONTHS BETWEEN 2nd and 3rd BIRTH

Degree of Intervention Indirect	Direct	Without Intervention	With indirect intervention	With direct interve-ntion	With direct and inter-vention
0.95	0.90	42.83	43 .23	45.09	45.49
0.95	0.85	42.83	43.23	46.43	46.82
0.95	0.80	42.83	43.23	47.93	48.32
0.95	0.75	42.83	43.23	49.64	50.03
0.95	0.70	42.83	43.23	51.60	51.99
0.90	0.85	42.83	43.68	45.60	45.93
0.90	0.80	42.83	43.68	46.09	47.26
0.90	0.75	42.83	43.68	47.43	48.76
0.90	0.70	42.83	43.68	49.64	50.47
0.90	0.70	42.83	43.68	51.60	52.43

Table 7.3 C
showing the change in the inter birth interval (3-4) as a result of different levels of intervention by Direct and Indirect Methods

BIRTH INTERVAL IN MONTHS BETWEEN THIRD AND FOURTH BIRTHS

Degree of Intervention Indirect	Direct	Without Intervention	With Indirect Intervention	With direct Intervention	With direct & Indirect Intervention
0.95	0.90	42.35	42.77	44.51	44.93
0.95	0.85	42.35	42.77	45.79	46.21
0.95	0.80	42.35	42.77	47.23	47.65
0.95	0.75	42.35	42.77	48.87	49.28
0.95	0.70	42.35	42.77	50.74	51.16
0.90	0.90	42.35	43.24	44.51	45.40
0.90	0.85	42.35	43.24	45.79	46.68
0.90	0.80	42.35	43.24	47.23	48.12
0.90	0.75	42.35	43.24	48.87	49.75
0.90	0.70	42.35	43.24	50.74	51.62

An overall glance in the Tables 5.3A, 5.3B and 5.3C reveal that inter conception (or inter birth) intervals substantially increases as a result of direct intervention whereas the indirect methods donot appear to be much effective. The same conclusion holds for all other order of conceptions also. Further in almost all the entries the joint linear effects of Direct and Indirect Interventions have been more or less found as the sum of the Direct and Indirect Interventions respectively showing the virtual absence of any interaction between the two types of interventions.

ON A GENERALISED PROBABILITY MODEL FOR MEASURING INTER CONCEPTIVE DELAYS

8.0 Measurement of Inter Conceptive delays is an important item of investigation in evaluating the fecundability status of a community. A correct appraisal of Inter Conceptive delays should take into consideration (1) variation of fecundability over different order of conceptions (ii) variability of fecundability between individuals at the same parity level.

The present exercise is devoted to develop systematically probability models concerning inter conceptive delays by taking accounts of both the factors into consideration. Counter theory of type I with fixed dead time has been employed to obtain the Palm Probabilities (viz. given that at a particular point of time a conception of a given order takes place the probability of the waiting time for the next conception after a fixed time t known as Palm Probability of obtaining the waiting time for the next conception. Estimation techniques concerning various parameters of the fertility process has also been attempted in the present exercise.

8.1 Notation and Development of the Model :

The problem is to find out the waiting time of $(i+1)^{th}$ conception given that the i^{th} conception has taken place at a fixed time, say T=0. (Palm Probability); subject to the condition that following i^{th} conception at T=0, an infecundable period consisting of the gestation period and the Post Partum Amenorrhoea follows, during which the probability of a reconception is Zero (π represents the fixed dead time in a Counter Model type I);

The waiting time distribution of $(i+1)^{th}$ conception given that the i^{th} conception has taken place at T=0 has been derived in Chapter 2 as

$$f(t) = \frac{a_i^{k_i+1}\,(k_i+1)}{(a_i+t)^{k_i+2}} \qquad i=1,2,\ldots\ldots \tag{8.1}$$

where (a_i, k_i) are the parameters for the $(i, i+1,)^{th}$ order of conceptions.

However the same result can more rigorously be derived using the p.g.f. as follows :

$V_k(t)$ = Probability of k events (conceptions in (o,t)) given that at T=0 an event (conception) has taken place (Palm Probability) and $\psi_k(t)$ = unconditional probability of k events in (O,t).

Then the probability generating functions (p.g.f.) $G(z,t)$ and $G_0(z,t)$ of $V_k(t)$ and $\psi_k(t)$ respectively are as follows:

$$G(z,t) = \sum_{k=0}^{\infty} V_k(t) z^k \quad \ldots.(8.2)$$

$$G_0(z,t) = \sum_{k=0}^{\infty} \psi_k(t) z^k \quad \ldots(8.3)$$

Then, from Chapter 4 it follows

$$G(z,t) = \left(\frac{a}{a+t(1-z)}\right)^k \quad \ldots.(8.4)$$

and

$$G_0(z,t) = \left(\frac{a}{a+t(1-z)}\right)^{k+1} \quad \ldots(8.5)$$

We have

$$\frac{\partial}{\partial z} G(z,x) = -\frac{K}{a}(1-z)\, G_0(z,x) \quad \ldots(8.6)$$

(vide Cox D.R. and Isham V. (1980), Point Processes, Chapman and Hall, London, p.30).

If T_n = time to n^{th} event given first one has happened at T=0. Then

$$F_n(t) = P(T_n \leq t) = \sum_{j=n}^{\infty} \psi_k(t) \quad \ldots(8.7)$$

$$H_0(z,t) = \sum_{n=0}^{\infty} z^n F_n(t) = \frac{1 - zG_0(z,t)}{1-z} \quad \ldots(8.8)$$

and if

$$f_n(t) = \frac{dF_n(t)}{dt} \qquad \text{...(8.9)}$$

be the density function corresponding to the $(n+1)^{th}$ arrival given that the first arrival is at T=0. Then

$$\sum_{n=0}^{\infty} z^n f_n(t) = \frac{-z \frac{\partial}{\partial t} G_o(z,t)}{1-z}$$

$$= \frac{z(k+1)}{a}\left(\frac{a}{a+t(1-z)}\right)^{k+2}$$

$$= \sum_{j=1}^{\infty} \frac{z^j a^{k+1} \Gamma(j+k+1)}{\Gamma(k+1)(a+t)^{k+j+1}(j-1)!}$$

derived in Chapter 2. ...(8.10)

Equating both sides the coefficients of z we get $f_1(t)$ viz. the waiting time distribution of the 2nd event given that the first has occurred at T=0.

Hence

$$f_1(t) = \frac{a^{k+1}\Gamma(k+2)}{\Gamma(k+1)(a+t)^{k+2}}$$

$$= \frac{a^{k+1}(k+1)}{(a+t)^{k+2}}$$

changing the parameters a,k to a_i, k_i for the (i+1) order of conception given that the i^{th} has taken place at T=0 we have

$$f_i(t) = \frac{a_i^{k_i+1}\,(k_i+1)}{(a_i+t)^{k_i+2}} \qquad \ldots(8.11)$$

$$0 \leq t < \infty.$$

Further if there is a dead time infecundable period π_i after T=0 then the waiting time distribution is given by

$$f_i(t|\pi) = \frac{a_i^{k_i+1}\,(K_i+1)}{(a_i+t-\pi_i)^{K_i+2}} \;;\; \pi \leq t < \infty \qquad \ldots(8.12)$$

The mean & 2nd raw moment from the origin of the distribution (6.12) is as follows :

$$\mu_1' = E_i(T|\pi) = \frac{a_i}{K_i} + \pi_i \qquad \ldots(8.13)$$

$$\mu_2' = \frac{2a_i^2}{K_i(K_i-1)} + \frac{2a_i\pi_i}{k_i} + \pi_i^2 \qquad \ldots(8.14)$$

Hence the probability of $(i+1)^{th}$ conception between $(t-1)^{th}$ to t^{th} month given that i^{th} event has taken place at T=0 (i=1,2,3.....) subject to the condition that there is an infecundable period π_i following T=0, given by

$$\int_{t-1}^{t} \frac{a_i^{k_i+1}\,(k_i+1)}{(a_i+t+\pi_i)^{k_i+2}}\,dt$$

$$= a_i^{k_i+1}\left[\frac{1}{(a_i+t)^{k_i+1}} - \frac{1}{(a_i+t+i)^{k_i+1}}\right] \qquad \ldots(8.15)$$

8.2 Estimation of the Parameters (a_r, k_r):

$r = 2,3,4,.....$

Let us consider $i = 1$ i.e. the parameters of the waiting time distribution of the second conception given that the first conception has taken place at T=O, subject to the condition that there is an infecundable period $_1$ following T=O. Using the same data, of Srinivasan (1972) as referred by Mitra and Banerjee (1982) estimating equations for k_1 and a_1 are as follows :

$$\mu_1' = 36.1 = \frac{a_1}{k_1} + 17.0 \qquad ...(8.16)$$

$$\mu_2' = 345.8 + (36.1)^2 = 1649.01$$

$$= \frac{2a_1^2}{k_1(k_1-1)} + \frac{2a_1(17)}{k_1} + (17)^2 \qquad ...(8.17)$$

Direct solutions by the method of moments are given as

$$\hat{k}_1 = 2.93804$$
$$\hat{a}_1 = 56.11664.$$

Similarly the using the mean and the second raw moment of the waiting time distribution between second and third order of conceptions as 38.9 and 1921.01 respectively and $\tau_2 = 18.2$. we have,

$$\hat{k}_2 = 2.90212$$
$$\hat{a}_2 = 60.07392.$$

Finally using μ'_1 = 38.6 and μ'_2 = 1870.06 months and π_3 = 18 6 months for the third and fourth order of conceptions we have

$$\hat{k}_3 = 3.01456$$
$$\hat{a}_3 = 60.29147.$$

The probability distributions of the women conceiving between $(t-t+1)^{th}$ months t = 0,1,2..... 59 for the $(i-i+1)^{th}$ order of conceptions (i = 1,2,3) assuming infecundable period π_i (i = 1,2,3) following the i^{th} conception is provided in table 6 respectively.

Table 8

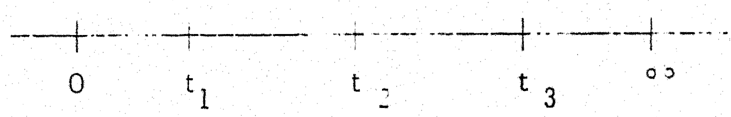
Probability distribution of waiting time of conception by months between different consecutive order of conceptions measured from the date of expiry of the infecundable period.

Month	Between First and second order of births	Between Second and Third order of births	Between Third and Fourth order of births
(1)	(2)	(3)	(4)
0-1	0.06719	0.06239	0.06391
1-2	0.06163	0.05758	0.05888
2-3	0.05661	0.05320	0.05433
3-4	0.05207	0.04922	0.05019
4-5	0.04796	0.04560	0.04642
5-6	0.04424	0.04229	0.04299
6-7	0.04086	0.03926	0.03986
7-8	0.03778	0.03650	0.03700
8-9	0.03498	0.03396	0.03438
9-10	0.03243	0.03163	0.03198
10-11	0.03009	0.02950	0.02977
11-12	0.02796	0.02753	0.02775
12-13	0.02600	0.02572	0.02589
13-14	0.02421	0.02405	0.02418
14-15	0.02256	0.02251	0.02260
15-16	0.02105	0.02109	0.02114
16-17	0.01966	0.01977	0.01980
17-18	0.01837	0.01855	0.01855
18-19	0.01719	0.01742	0.01740
19-20	0.01609	0.01638	0.01634
20-21	0.01508	0.01540	0.01535
21-22	0.01415	0.01450	0.01443
22-23	0.01328	0.01366	0.01358
23-24	0.01248	0.01288	0.01278
24-25	0.01173	0.01215	0.01204
25-26	0.01104	0.01147	0.01136
26-27	0.01039	0.01083	0.01072
27-28	0.00979	0.01024	0.01012
28-29	0.00924	0.00968	0.00956
29-30	0.00872	0.00917	0.00904

(1)	(2)	(3)	(4)
30-31	0.00823	0.00868	0.00855
31-32	0.00778	0.00823	0.00809
32-33	0.00735	0.00780	0.00766
33-34	0.00696	0.00740	0.00726
34-35	0.00658	0.00702	0.00689
35-36	0.00624	0.00667	0.00653
36-37	0.00591	0.00634	0.00620
37-38	0.00561	0.00603	0.00589
38-39	0.00532	0.00573	0.00560
39-40	0.00505	0.00546	0.00532
40-41	0.00480	0.00519	0.00506
41-42	0.00456	0.00495	0.00482
42-43	0.00434	0.00472	0.00459
43-44	0.00412	0.00450	0.00437
44-45	0.00393	0.00429	0.00416
45-46	0.00374	0.00410	0.00397
46-47	0.00356	0.00391	0.00379
47-48	0.00340	0.00374	0.00361
48-49	0.00324	0.00357	0.00345
49-50	0.00309	0.00341	0.00330
50-51	0.00295	0.00326	0.00315
51-52	0.00282	0.00312	0.00301
52-53	0.00269	0.00299	0.00288
53-54	0.00257	0.00286	0.00276
54-55	0.00246	0.00274	0.00265
55-56	0.00235	0.00263	0.00252
56-57	0.00225	0.00252	0.00242
57-58	0.00216	0.00242	0.00232
58-59	0.00206	0.00232	0.00222
59-60	0.00198	0.00222	0.00213
60 +	0.05707	0.06705	0.06250

A SURVIVAL THEORY APPROACH FOR OBTAINING THE DURATION OF INTER LIVE BIRTH INTERVALS

9.0 Here we assume initially increasing and then decreasing hazard rate for i^{th} order of conception, $(i=1,2,3,\ldots.)$ during the open interval $[0,\infty)$; where the origin of time starts from the beginning of the fecundable period for the i^{th} parity (after the termination of the infecundable period (post partum amenorrhoea) corresponding to the $(i-1)^{th}$ parity.) We assume a one to one correspondence between live birth and conception for the sake of simplicity.



Let the hazard rate following the termination of $(i-1)^{th}$ birth $(i=1,2,\ldots\ldots\ldots\ldots,n)$ be given as

$$\left.\begin{aligned} \lambda_i(t) &= \lambda_i, & 0 \le t < t_1 \\ &= \lambda_i e^{\delta_i}, & t_1 \le t < t_2 \\ &= \lambda_i e^{\delta_i'}, & t_2 \le t < t_3 \\ &= \lambda_i e^{\delta_i'}, & t \ge t_3 \end{aligned}\right\} \ldots (9.1)$$

where $\lambda_i > 0$ and $\delta_i > 0$, $\delta_i' < 0$; $, 0 < \lambda_i < \infty$

If we denote the survival function $R_i(t)$ given by

$$R_i(t) = P(T_i \geq t) = 1-F_i(t)$$

then

$$
\left.
\begin{aligned}
R_i(t) &= e^{-\lambda_i t}, \qquad 0 \leq t < t_1 \\
&= e^{-\lambda_i t_1}\, e^{-\lambda_i (t - t_1) e^{\delta_i'}}, \qquad t_1 \leq t < t_2 \\
&= e^{-\lambda_i t_1}\, e^{-\lambda_i (t_2 - t_1) e^{\delta_i} - \lambda_i e^{\delta_i'} (t - t_2)} \\
&\qquad\qquad t_2 \leq t < t_3 \\
&= e^{-\lambda_i t_1}\, e^{-\lambda_i e^{\delta_i} (t_2 - t_1)}\, e^{-\lambda_i e^{\delta_i'} (t_3 - t_2)}\, e^{-\lambda_i e^{\delta'} (t - t_3)} \\
&\qquad\qquad \text{for } t \geq t_3
\end{aligned}
\right]
$$

...(9.2)

$$E(T_i) = \int_0^{\infty} R(t)dt \qquad ...(9.3)$$

$$Var(T_i) = \int_0^{t} t \ R(t)\ dt - \left(\int_0^{\infty} R(t)\ dt\right)^2 \qquad ...(9.4)$$

and the moments of k^{th} order ($k = 1,2,3......$) from the origin is given by

$$E(T_i^k) = \left(k\int_0^{\infty} t^{k-1} R(t)dt\right). \qquad ...(9.5)$$

$k = 1,2,3......$

9.1 Estimation of the Parameters

Let n_1, n_2, n_3, and n_4 be the mothers who had the i^{th} order of conception (and a live birth) in $[0,t_1)$, $[t_1,t_2)$, $[t_2,t_3)$ and $[t_3,\infty)$ respectively. Also the times of having conception of the 1st, 2nd, n_ith mother during $[t_{j-1},t_j)$; $j=1,2,3,4$ $t_0=0$, $t_4\to\infty$ be recorded as $t_{1i}, t_{2i}, \ldots\ldots, t_{ni}$ respectively. We further denote the likelihood function of the sample as L_1, L_2, L_3 and L_4 respectively.

Then

$$L_1 = \frac{n!}{n_1!(n-n_1)!}\lambda_i^{n_1} e^{-\lambda_i(t_{1i}+t_{2i}+\cdots+t_{ni})} \quad \text{subject to } t_{ji} \le t_i,\ j=1,2\cdots n_1$$
$$(e^{-\lambda_i t_1})^{n-n_1}$$

$$\Rightarrow \frac{\partial \log L_1}{\partial \lambda_i} = \frac{n_1}{\lambda_i} - \sum_{j=1}^{n_1} t_{ji} - (n-n_1)t_j = 0 \qquad \ldots(9.6)$$

$$\Rightarrow \hat{\lambda}_i = \frac{n_1}{\left[\sum_{j=1}^{n_1} t_{ji} + (n-n_1)t_j\right]} \qquad \ldots(9.7)$$

Precisely in a similar way,

$$L_2 = \frac{(n_2+n_3+n_4)!}{n_2!(n_3+n_4)!}(e^{-\lambda_i t_1})^{n_2}(\lambda_i e^{\delta_i})^{n_2}$$
$$e^{-\lambda_i e^{\delta_i}\cdot\sum_{j=1}^{n_2} t_{2j}}$$
$$(e^{-\lambda_i e^{\delta_i} t_2})^{\underbrace{n-n_1-n_2}_{=n_3+n_4}} \qquad \ldots(9.8)$$

$$\Rightarrow \operatorname{Log} L = \operatorname{Log} C_2 + n_2(-\lambda_i t_1) + n_2[\log \lambda_i + \delta_i] - \lambda_i e^{\delta_i} \sum_{j=1}^{n_2} t_{2j} + (n_3+n_4)(-\lambda_i e^{\delta_i} t_2) = 0$$

$$\Rightarrow \frac{\partial \log L}{\partial \lambda_i} = -n_2 t_1 + \frac{n_2}{\lambda_i} - e^{\delta_i} \sum_{j=1}^{n_2} t_{2j} - (n_3+n_4) e^{\delta_i} t_2 = 0$$

$$\Rightarrow \frac{\lambda_i}{n_2} = \frac{1}{n_2 t_1 + e^{\delta_i} \sum_{j=1}^{n_2} t_{2j} - (n_3+n_4) e^{\delta_i} t_2}$$

$$\Rightarrow \hat{\lambda}_i = \frac{n_2}{n_2 t_1 + e^{\delta_i} \sum_{j=1}^{n_2} t_{2j} - (n_3+n_4) e^{\delta_i} t_2} \quad ..(9.9)$$

on substitution of the estimate of λ_i from (9.7) in (9.9) one can get the m.l.e. of δ_i.

Finally to estimate δ_i' we get

$$r_3 = \frac{(n_3+n_4)!}{n_3!\, n_4!} \left[e^{-\lambda_i t_1} e^{-\lambda e^{\delta_i}(t_2-t_1)} \right]^{n_3}$$

$$(e^{-\lambda_i t_1})^{n_4} \left(e^{-\lambda_i e^{\delta_i}(t_2-t_1)}\right)^{n_4}$$

$$\left(e^{-\lambda_i e^{\delta_i'}(t_3-t_2)}\right)^{n_4}$$

$$\cdot (\lambda_i e^{\delta_i'})^{n_3} e^{-\lambda_i e^{\delta_i'}(t_{31}+t_{32}+\cdots+t_{3n_3})}$$

$$\mathrm{Log}\, L_3 = \mathrm{Log}\, c + n_3\left[-\lambda_i t_1 - \lambda_i e^{\delta_i}(t_2-t_1)\right]$$
$$+ n_3\left[(\log\lambda_i + \delta_i')\right] - \lambda_i e^{\delta_i'}(t_{31}+t_{32}+\cdots+t_{3n})$$
$$+ n_4\left[-\lambda_i t_1 + (-\lambda_i e^{\delta_i}(t_2-t_1))\right] + n_4\left[-\lambda_i e^{\delta_i'}(t_3-t_2)\right]$$

$$\frac{\delta \log L_3}{\delta \lambda_i} = -n_3 t_1 - n_3 e^{\delta_i}(t_2-t_1) - \frac{n_3}{\lambda_i}\delta_i'(t_{31}+t_{32}+\cdots+t_{3n})$$
$$- e^{\delta_i'}(t_{31}+t_{32}+\cdots+t_{3n})$$
$$- n_4 t_1 - n_4 e^{\delta_i}(t_2-t_1) - n_4 e^{\delta_i'}(t_3-t_2) = 0$$

....(9.10)

on substitution of the estimated values of λ_i and δ_i from (9.7) & (9.9) in (9.10) we get the m.l.e. of δ_i'. On substitution of the estimated values of λ_i, δ_i and δ_i', we get the estimates of the interlive birth interval between $(i-1)^{th}$ parity and the i^{th} prity (i = 1,2,3,.....) given by

$$E(T_i) = \int_0^{\infty} R_i(t)\, dt$$

$$= \left[\int_0^{t_1} \lambda_i e^{-\lambda_i t}\, dt + \int_{t_1}^{t_2} e^{-\lambda_i t_1 - \lambda_i e^{\delta_i}(t-t_1)}\, dt + \right.$$

$$+\int_{t_2}^{t_3} e^{-\lambda_i t} e^{-\lambda_i e^{\delta_i(t_2-t_1)}} e^{-\lambda_i e^{\delta_i'(t-t_2)}} dt$$

$$+\int_{t_3}^{\infty} e^{-\lambda_i t} e^{-\lambda_i e^{\delta_i'(t_3-t_2)}} e^{-\lambda_i e^{-2\delta_i'(t-t_4)}} dt \Big],$$

$$= \frac{1}{\lambda_i}\left[1-e^{-\lambda_i t_1}\right] + \left[\frac{e^{-\lambda_i t_1}\left(1-e^{-\lambda_i e^{\delta_i(t_2-t_1)}}\right)}{\lambda_i e^{\delta_i}}\right]$$

$$+\left[\frac{e^{-\lambda_i t_i} e^{-\lambda_i e^{\delta_i(t_2-t_1)}} \cdot \left(1-e^{-\lambda_i e^{\delta_i'(t_3-t_2)}}\right)}{\lambda_i e^{\delta_i'}}\right] \quad \ldots(9.11)$$

While the variance of the waiting time is given by

$$Var(T_i) = 2\Big[\int_0^{t_1} t \cdot e^{-\lambda_i t} dt$$

$$+\int_{t_1}^{t_2} t\, e^{-\lambda_i t} e^{-\lambda_i e^{\delta_i}} dt$$

$$+\int_{t_2}^{t_3} t\, e^{-\lambda_i t} e^{-\lambda_i e^{\delta_i}} e^{-\lambda_i e^{\delta_i'}} dt$$

$$+\int_{t_3}^{\infty} t\, e^{-\lambda_i t} e^{-\lambda_i e^{\delta_i(t_2-t_1)}} e^{-\lambda_i e^{\delta_i'(t_3-t_2)}} \cdot e^{-\lambda_i 2\delta_i'(t-t_2)(t-t_3)} \cdot dt$$

$$-\left(E(T_i)\right)^2$$

or $\Rightarrow$

$$
\begin{aligned}
Var(T_i) = \frac{2}{\lambda_i^2}\Big\{\Big[\,[(1-e^{-\lambda_i t_1}(1+\lambda_i t_1))\\
&+ e^{-\lambda_i t_1}(1+\lambda_i e^{\delta t_1} - (1+\lambda_i e^{\delta_i t_2})(e^{-\lambda_i e^{\delta_i (t_2-t_1)}})\Big/ e^{2\delta_i}\\
&+ \frac{e^{-\lambda_i t_1}\, e^{-\lambda_i e^{\delta_i (t_2-t_1)}}\,(1+\lambda_i e^{\delta_i' t_2} - (1+\lambda_i e^{\delta_i' t_3})\, e^{-\lambda_i e^{\delta_i' (t_3-t_2)}}}{e^{2\delta'}}\Big\}\\
&+ \frac{e^{-\lambda_i t_1}\, e^{-\lambda_i e^{\delta_i (t_3-t_1)}}\, e^{-\lambda_i e^{\delta_i (t_3-t_2)}}\{1+\lambda_i e^{\delta_i' t_3}\}}{e^{2\delta_i'}}\\
&- (E\,(T_i))^2 \qquad \ldots(9.12)
\end{aligned}
$$

9,2 A Numerical Illustration

We begin with some hypothetical data, and using the same to obtain the estimated ~~main~~ interval between 1st and 2nd conception

Let $i = 2$; $N = n_1 + n_2 + n_3 + n_4 = 63$

$t_1 = 1$, $t_2 = 2$, $t_3 = 3$ (years)

$n_1 = 25$, $n_2 = 8$, $n_3 = 30$, $n_4 = 0$.

By sufficiently extending t_3 till the end of the fertility span it is possible to make $n_4 = 0$ and thus avoid the problem of censoring in the estimates procedure.

Let,

$$\sum_{j=1}^{n_1} t_{1j} = 25 \qquad \sum_{j=1}^{n_2} t_{2j} = 16, \qquad \sum_{j=1}^{n} t_{3j} = 300$$

Then, $\hat{\lambda}_1 = 0.35$, $\hat{\delta}_1 = 0.3168$, $\hat{\delta}'_1 = -1.5633$.

$E(T_1) = 2.3415$ (years)

$Var(T_1) = 21.0987$.

s.t. error of $T_1 = \dfrac{\sqrt{Var(T_1)}}{\sqrt{63}} = 0.5787$.

The normal confidence limit of the waiting time between 1st to 2nd conception lies between 0.6054 to 4.0776 with more than 99% level of probability.

9.3 Conclusion

(i) This simple method of maximum likelihood estimation of interlive birth interval may be employed over all parities and then expected waiting time over parities may be compared, to show the fertility behaviour.

(ii) The model admits easy generalisation in the case of the assumption of one to one correspondence between conception and live birth is dropped.

(iii) The model avoids the problem of censoring & complicated estimation techniques thereof.

(iv) The model admits further generalisation if individuals variation is fecundity within the same parity is considered: although the result remain more complicated.

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A GENERALISATION OF FREUND'S BIVARIATE EXPONENTIAL MODEL FOR INTERRELATION OF POST PARTUM AMENORRHOEA & LACTATION

10.0 A wealth of earlier empirical studies have revealed that prolonged breast feeding following a child birth is helpful in postponing the date of resumption of fresh ovulatory cycle by prolonging the period of post partum amenorrhoea. Given that lactation has been discontinued at some time following a child birth at age t of the mother, the problem is to obtain the residual life time of the period of post partum amenorrhoea (or infecundable period). This problem is of considerable importance in family limitation programme while studying how far the encouragement of the lactation process is helpful in preventing reconception. The problem is again similar to one in survival analysis of paired organs, viz., if out of two kidneys (parallel system), one fails the residual survival time of the second kidney is associated with higher hazard rate and hence its longevity is reduced.

10.1 This kind of representation is provided in Freund's (1961) Bivariate exponential model given by

$$f(x,y) = \alpha\beta' e^{-\beta' y-(\alpha+\beta-\beta')x} \qquad 0 < x < y \qquad \ldots(10.1)$$

$$= \beta\alpha' e^{-\alpha' x-(\alpha+\beta-\alpha')y} \qquad 0 < y < x \qquad \ldots(10.2)$$

Denoting lactation period (L.P.) by X and post partum amenorrhoea (P.P.A) by Y and α and β being the intensities of expiring of L.P and P.P.A respectively subject to the condition that on expiry of lactation the hazard rate for expiry of P.P.A. is changed from β to β' and on expiry of P.P.A. first the hazard rat^ of expiry of lactation period is changed from α to α'. Usually $\alpha' \geq \alpha$ and $\beta' \geq \beta$.

The representation of (10.1) and (10.2) however do not take into account of the age of the mother. We replace α by α(t) and β by β(t) where α(t) and β(t) both being functions of (α ,t) and (β ,t) respectively without α and β being dependent on t.

We can write

$$f(x,y \mid t) = \alpha(t)\,\beta'(t)\,e^{-\beta'(t)y-(\alpha(t)+\beta'(t)-\beta'(t))x} \qquad 0<x<y \qquad \ldots(10.3)$$

$$= \beta(t)\,\alpha'(t)\,e^{-\alpha'(t)x-(\alpha(t)+\beta(t)-\alpha'(t))y}$$

$$0< y < x \qquad \ldots(10.4)$$

A specific form say

$$\alpha(t) = \alpha t^{\alpha-1} \qquad \alpha \gtrless 1$$

and $$\beta(t) = \beta t^{\beta-1} \qquad \beta \gtrless 1 \qquad \ldots(10.5)$$

which correspond to Weibull failure rate. A specific example of the application (10.3) and (10.4) lies in finding out the conditional probability of expiry of L.P. and P.P.A. at the age of (t+x) and (t+y) respectively when (x < y); or P.P.A. and L.P. at the ages (t+y) and (t+x) respectively when y < x.

10.2 Moments of the distribution

$$M(\theta_1, \theta_2 \mid t) = E(e^{\theta_1 X + \theta_2 Y} \mid t)$$

$$= \int_0^\infty \int_x^\infty e^{\theta_1 x + \theta_2 y} \alpha(t)\beta'(t) e^{-\beta'(t)y - (\alpha(t) + \beta(t) - \beta'(t))x} \, dx\, dy$$

$$+ \int_0^\infty \int_y^\infty e^{\theta_1 x + \theta_2 y} \alpha'(t)\beta(t) e^{-\alpha'(t)x - (\alpha(t) + \beta(t) - \alpha'(t))y} \, dx\, dy$$

$$\ldots(10.6)$$

Routine calculation provides

$$M(\theta_1,\theta_2|t) = [(\alpha(t)+\beta(t))^{-1}((\alpha(t)+\beta(t))$$

$$+\frac{\alpha(t)\theta_2}{\beta'(t)}+\frac{\alpha(t)\theta_2^2}{(\beta'(t))^2}+\frac{\beta(t)\theta_1}{\alpha'(t)}+\frac{\beta(t)\theta_1^2}{(\alpha'(t))^2}$$

$$+\frac{\alpha(t)(\theta_1+\theta_2)}{\alpha(t)+\beta(t)}+\frac{\alpha(t)\theta_2}{\beta'(t)}+\frac{\theta_1+\theta_2}{\alpha(t)+\beta(t)}$$

$$+\frac{\alpha(t)\theta_2^2}{(\beta'(t))^2}\cdot\frac{\theta_1+\theta_2}{\alpha(t)+\beta(t)}+\beta(t)\frac{\theta_1+\theta_2}{\alpha(t)+\beta(t)}$$

$$+\frac{\beta(t)\theta_1}{\alpha'(t)}\frac{\theta_1+\theta_2}{\alpha(t)+\beta(t)}+\frac{\beta(t)\theta_1^2}{(\alpha'(t))^2}\frac{\theta_1+\theta_2}{\alpha(t)+\beta(t)}$$

$$+\alpha(t)\frac{\theta_1+\theta_2}{(\alpha(t)+\beta(t))^2}+\frac{\alpha(t)\theta_2}{(\beta'(t))^2}\cdot\frac{(\theta_1+\theta_2)^2}{(\alpha(t)+\beta(t))^2}$$

$$+\frac{\alpha(t)\theta_2^2}{(\beta'(t))^2}\frac{(\theta_1+\theta_2)^2}{(\alpha(t)+\beta(t))^2}+\beta(t)\frac{(\theta_1+\theta_2)^2}{(\alpha(t)+\beta(t))^2}$$

$$+\frac{\beta(t)\theta_1}{\alpha'(t)}\cdot\frac{(\theta_1+\theta_2)^2}{(\alpha(t)+\beta(t))^2}+\frac{\beta(t)\theta_1^2(\theta_1+\theta_2)^2}{(\alpha'(t))^2(\alpha(t)+\beta(t))^2}+$$

$$\ldots\ldots\ldots\ldots\ldots)) \qquad \ldots(10.7)$$

$\Rightarrow$

$$E(X|t) = (\alpha(t) + \beta(t))^{-1} \left(\frac{\beta(t)}{\alpha'(t)} + \frac{\alpha(t)}{\alpha(t)+\beta(t)} + \frac{\beta(t)}{\alpha(t)+\beta(t)}\right)$$

= Coefficient of θ_1 in the expansion of $M(\Theta_1, \Theta_2| t)$

$$= \frac{\alpha'(t)+\beta(t)}{\alpha'(t)(\alpha(t)+\beta(t)} \quad \ldots(10.8)$$

Similarly

$$E(X^2|t) = \frac{2}{\alpha(t)+\beta(t)} \cdot \frac{(\beta(t)(\alpha(t)+\beta(t))+\alpha'(t)\beta(t)\beta(t)+\alpha'(t))^2)}{(\alpha'(t))2 \quad (\alpha(t)+\beta(t))}$$

...(10.9)

$$E(Y|t) = \frac{\alpha(t)+\beta'(t)}{\beta'(t)(\alpha(t)+\beta(t)} \quad \ldots(10.10)$$

$$E(Y^2|t) = \frac{2}{(\alpha(t)+\beta'(t)} \quad \frac{(\alpha(t)(\alpha(t)+\beta(t)+\beta'(t)\alpha(t)-\beta'(t))^2)}{(\beta'(t))2 \quad (\alpha(t)+\beta(t)}$$

...(10.11)

$E(XY|t)$ = Coefficient of $\Theta_1 \Theta_2$ in the expansion of $M(\Theta_1, \Theta_2| t)$

$$= \frac{\alpha(t)\alpha'(t)+\beta(t)\beta'(t)+2\alpha'(t)\beta'(t)}{(\alpha(t)+\beta(t))^2 \ \alpha'(t)\beta'(t)} \quad \ldots(10.12)$$

$\Rightarrow$

$$Var(X|t) = E(X^2|t)-(E(X|t)^2$$

$$= \frac{(\alpha'(t))^2+ 2\alpha(t)\beta(t)+(\beta(t))^2}{(\alpha'(t))^2(\alpha(t)+\beta(t))^2} \quad \ldots(10.13)$$

$$\mathrm{Var}(Y|t) = \frac{(\alpha(t))^2 + 2\alpha(t)\beta(t) - (\beta'(t))^2}{(\beta'(t))^2 \; (\alpha(t)+\beta(t))^2} \qquad \ldots(10.14)$$

$$\mathrm{Corr}(X,Y|t) = \frac{E(XY|t) - E(X|t).E(Y|t)}{\sqrt{\mathrm{Var}(X|t)}\,\sqrt{\mathrm{Var}(Y|t)}}$$

$$= \frac{\alpha'(t)\,\beta'(t) - \alpha(t)\,\beta(t)}{[(\alpha'(t))^2 + 2\alpha(t)\beta(t) + (\beta(t))^2][(\beta'(t))^2 + 2\alpha(t)\beta(t) + (\alpha(t))^2]} \qquad \ldots(10.15)$$

10.3 Estimation of the parameters by the method of moments under the case $\alpha(t) = \alpha t^{\alpha'-1}$ and $\beta(t) = \beta t^{\beta'-1}$.

Given t the age of the mother at the child birth and using (10.8), (10.10), (10.13) and (10.14) for $E(X|t)$, $E(Y|t)$, $\mathrm{Var}(X|t)$ and $\mathrm{Var}(Y|t)$ one can get the estimates of α, β, α' and β' at different levels of t say t = 15, 20, 25, 30, 35 and 40. One may combine the estimates assuming that the parameters viz α, β, α' and β' are independent of the age t of the mother. Then using (10.15), one can get the correlation ρ_{XY} between L.P. and P.P.A. An approximate linear regression is given by

$$\left[Y - E(Y|t)\right] = \rho_{XY} \frac{\sqrt{\mathrm{Var}(Y|t)}}{\sqrt{\mathrm{Var}(X|t)}} \left[(x - E(X|t))\right]$$

or

$$Y = E(Y|t) + \rho_{XY} \frac{\sqrt{Var(Y|t)}}{\sqrt{Var(x|t)}} (x-E(X|t) \qquad ...(10.16)$$

and

$$x = E(X|t) + \frac{\rho_{XY}\sqrt{Var(X|t)}}{\sqrt{Var(Y|t)}} (Y-E(Y|t) . \qquad ...(10.17)$$

(10.16) will thus predict the period of P.P.A. given L.P. when L.P. < P.P.A.

(10.17) will predict the period of L.P. given P.P.A. when P.P.A. < L.P.

A numerical illustration of the technique

The purpose ofthe Freund's model is to predict the Post Partum amenorrhoea (PPA) given the lactation period and lactation period given the PPA. These may be done by using the results (10.16) and (10.17) respectively by substituting the estimates of the parameters i.e. α (in $\alpha(t) = \alpha t^{\alpha-1}$ and), β (in $\beta(t) = \beta t^{\beta-1}$ and for ~~α'~~ and β' respectively a similar process. Then by by the method of moments given the sample mean $\hat{E}(X|t)$), $\hat{E}(Y|t)$, $\widehat{Var}(X|t)$ and $\widehat{Var}(Y|t)$ at different levels of t.

Let us take the hypothetical example giving

$\hat{E}(X|t) = 1.89$ $\qquad$ $\hat{E}(Y|t) = 2.60$

$\widehat{Var}(X|t) = 3.08$ $\qquad$ $\widehat{Var}(Y|t) = 5.65$ at any $t = 15$

using the equations (10.8), (10.10), (10.13) and (10.14) the estimates of the parameters α, β, α' and β' are obtained by successive iterations as follows :-

$\hat{\alpha} = 0.46$, $\hat{\beta} = 0.31$

$\hat{\alpha}' = 0.68$ $\hat{\beta}' = 0.46$ at $t = 15$

Putting these estimates in (10.15) we get

$\widehat{\text{Cor}}(X,Y|t)$ (t = 15)

$= 0.2202$

on substitution of $\hat{E}(X|t)$, $\hat{E}(Y|t)$

$\widehat{\text{Var}}(X|t)$, $\widehat{\text{Var}}(Y|t)$ and

$\widehat{\text{Corr}}(X,Y|t)$ in (10.16) and (10.17)

respectively we get the prediction of Y (P.P.A) given X = 3 at t = 15.

$\Rightarrow$

Y = 2.89 months,

Conclusion

Precisely one may obtain by the same way the estimated prediction of X(lactation period) given Y=6.00(P.P.A). However, the

examples are too hypothetical to illustrate further application of the model based on real data. It is in fact, difficult to obtain the empirical joint distribution of P.P.A. and lactation period. Besides that the problem of censoring is also present there even a follow up is made after a long gap. Following a birth Under these difficulties, it is possible to employ the model to predict P.P.A. given lactation and vice and versa. However, the real efficacy of the methodology, can be understood when more extensive live data are obtained.

CONCLUSIONS

The studies in this dissertation pertain to stochastic modelling of population in the area of fertility analysis and family planning programmes. The study begins with a technique based on Palm Probability for the comparison of fertility statuses of two otherwise socio-economically identical groups, but differing by the age of effective marriage. The idea is to examine the policy of reduction of fertility by postponement of marriage. The solution which has been derived is purely methodological one applied on hypothetical data. Nevertheless, it is believed that the methodology developed may be tested on a more exhaustive set of real data to bring out more meaningful result. Similarly in chapter four a model based on Palm Probability, so developed has been employed to obtain the waiting time distribution of first conception from marriage and waiting time distribution between consecutive order of conceptions.

The traditional model building exercise has been modified here by using Palm Probabilistic technique. But the application of the technique has been done while keeping restriction of infinite marital exposure. In this connection comparison of the results based upon the approach of infinite marital exposure with that of finite marital

exposure (Srestha and Biswas (1985)) reveals that the gain in precision in the estimate of interconception intervals for lower order of conceptions is almost negligible. However, for higher order conceptions further investigations are necessary in this regard.

Attemps have been made in chapter five to develop a Multistage Markov Chain Model based on a density dependent birth and death process to obtain the asymtotic probability of having fixed number of surviving children based upon the same distribution. Attempts have also been made to decide an optimal sterilization policy depending on the number of surviving children.

One major indication in this study, although a methodological one based on hypothetically chosen fertility and mortality rates, is that if the policy is to ensure at least one surving male child in the long run (i.e. it is essentially desirable lower limit in India) then the result shows that irrespective of the variation in the levels of fertility and mortality the level of sterilization may be made on the basis of three surviving children. If policy is relaxed to the extent of sterilizing mothers only on the basis of more than three surviving children, then one may expect higher growth rate which may not be consistent with the policy of reduction of growth rate. Finally, if an intermediate action based on the policy of two surviving children is adopted then the condition of ensuring one surviving

male child in the long run per couple may only be realised with very low probability. Although the same may fulfil the plan of reducing the growth rate, it may not be culturally accepted by the people at large.

In chapter six several martingales have been constructed on the continuous parameter birth and death process to obtain the stopping time for sterilizing a mother and also estimating the proportion of eligible mothers for sterilization in future, given that each of them has a fixed number of children at present.

This study is based on hypothetical data. Nevertheless, this kind of methodological investigation is useful in answering two basic problems. The first one relates to cost of the sterilization programme. The percentage of mothers obtained for the sterilization programme based on the choice of different number of surviving children is indicative of the increase or decrease of the probable cost if we switch over a more liberal or more conservative type of policy while adopting the sterilization programme. The second one relates to the stopping time i.e., one can indirectly calculate the age of the mother at the time of sterilization connected with various sterilization programme. Further investigation of these two items of consideration may be of crucial importance in our national planning. The appliction of Martingale theory in such problems with live data would indeed be interesting.

Chapter seven deals with the evolution of direct and indirect strategies for reduction of conception rate based on an approach of Geiger Muller Counter Model. The direct strategies correspond to that of mechanical methods of family planning measures, whereas the indirect strategies lead to lengthening the period of lactational amenorrhoea and abstinence period (i.e. temporary seperation of husbands and wives from conjugal life following a child birth) as it is necessitated by social customs in several communities. The methodology has been applied on Srinivasan's data (1972) based on the experience of a sample of South Indian Women but the rates of conception as it may exist under the assumption of direct and indirect strategies of family limitation programmes on both are again based on hypothetical data. As such the investigation again becomes a methodological one. Nevertheless such investigation may highlight the rate of both direct and indirect strategies of family limitation in reducing the birth rate. The relative efficacies of the factors are to be taken into consideration for evolving suitable family planning programme.

In Chapter eight a generalised probability model for measuring inter-conceptive delays have been attempted. Inter-conceptive delays inthis study have been taken as the function of order of conception as well as individual fecundability affected by biological

and social condition. The non-susceptible period between conceptions have been accounted by the dead time of the counter model and the analysis has been attempted to be more precise by considering the probability of a conception based upon previous fertility behaviour. Further application with real data may be fruitful for this kind of methodological exercise.

Chapter nine deals with the survival theory approach for obtaining the inter live birth interval; based upon a survival function which has been chosen to be quite flexible i.e., it can take into account any type of fertility behaviour, viz., increasing fertility with age, decreasing fertility with age or a fertility function showing increasing as well as decreasing fertility behaviour at different age segments.

The estimation of the parameters of the model by the method of maximum likelihood is straight forward. This has been illustrated while taking the hypothetical data concerning the first & the second order of conceptions. However, the same can be easily generalised for higher order of conceptions. One advantage as already has been considered in this kind of approach is that it avoids the problem of censoring. Even one to one correspondence between live birth and conception as has been assumed all along in the exercises is not essential for building up a generalised model. However

with such realistic assumptions mathematical complications would be quite considerable but this would be indeed a worth taking exercise. Further exploration of the exact status of the maximum likelihood estimator in this case may also be very useful.

Finally chapter ten deals with a study of the role of lactation period in affecting the period of Post-partum amenorrhoea. The novelty of the methodological approach lies in generalizing Freund's model which is a bivariate exponential model with certain degree of dependence between the time of failure of two components or two sources of failure subject to the age dependence of the failures; where the failures are defined as the period of completion of the Post Partum Amenorrhoea and stoppage of lactation period of mothers respectively.

Post partum amenorrhoea is a natural factor but with the cessation of the same the lactation period is discontinued with greater intensity interestingly comparable with that of the time of failure of a second component given that the first component has already failed in a bilateral parallel system. Further, demographically speaking the problem has become more complicated because of the dependence of the two factors being influenced by the age of the mother. A generalisation of Freund's model has been found to be quite fruitful tool to analyse

the results. However, the limitation of the study is hypothetical data. A meaningful conclusion on the applicability of Freund's model as well as the result based on the same would be an useful item of information based upon exhaustively collected real data.

The application of Palm Probability, Counter Model, Renewal Theory and other stochastic process oriented techniques as Martingale Theory are some new methodological applications for arriving at solution of ferility analysis. However the limitation ofthe study is that all the studies are methodological and all solutions have been evolved only on theoretical plane but the applications of the newly evolved techniques and their justification will much depend naturally on more exhaustive type of real data. This is the motivation of making the methodological exercise for a fruitful application in the area of fertility investigation.

APPENDIX

To show

$$E(n) = (k+1)a^{k+1}\ A'(T-t_1)\ \left(\frac{A(T-t_1)}{a+A(T-t_1\ )}\right)$$

$$\left(1 - \frac{A(T-t_1)}{a+A(T-t_1)}\right)^{-(k+2)} \qquad \ldots(A)$$

and

$$Var\ (n) = \frac{a^{k+1}(k+1)\ A'\ (T-t_1)\ A\ (T-t_1)}{(a+A\ (T-t_1))} \left(\frac{a}{a+A(T-t_1)}\right)^{-(k+2)}$$

$$+\ a^{k+1}\ A'\ (T-t_1)\ (k+1)\ (k+2)\left(\frac{A(T-t_1)}{a+A\ (T-t_1)}\right)^2$$

$$\left(\frac{a}{a+A\ (T-t_1)}\right)^{-(k+3)}$$

$$-\left[(k+1)a^{k+1}\ A'\ (T-t_1)\ \left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)\right.$$

$$\left.\left(1-\frac{A(T-t_1)}{a+A\ (T-t_1)}\right)^{-(k+2)}\right]^2 \qquad \ldots(B)$$

given that

$$\emptyset_n\ (t) = \frac{(k+1)\ \ldots\ldots\ldots(k+n)}{n!}\ a^{k+1}\ A'(T-t_1)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^n.$$

We have

$$E(n) = \sum_{n=0}^{\infty} n\emptyset_n\ \ (T-t_1)$$

$$= \sum_{n=0}^{\infty} \frac{n(k+1)\,\ldots\ldots\ldots(k+n)}{n!} a^{k+1} \; A'(T-t_1)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^n$$

$$= \sum_{n=0}^{\infty} \frac{(k+1)\ldots\ldots\ldots\ldots(k+n)}{(n-1)\,!} a^{k+1} A' \; (T-t_1)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^n$$

$$= a^{k+1} \; A' \; (T-t_1)\Big((k+1)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)$$

$$+ \frac{(k+1)(k+2)}{1!}\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^2$$

$$+ \frac{(k+1)\;(k+2)\;(k+3)}{2!}\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^3 \quad +\ldots\ldots\ldots$$

$$= a^{k+1} \; A'(T-t_1) \; (k+1)\frac{A(T-t_1)}{a+A(T-t_1)}\Big(1+ \; \frac{(k+2)}{1!} \; \frac{A(T-t_1)}{a+A(T-t_1)}$$

$$+ \frac{(k+2)\;(k+3)}{2!}\left(\frac{A\;(T-t_1)}{a+A\;(T-t_1)}\right) \quad +\ldots\ldots\ldots\ldots\Big)$$

$$= \frac{(k+1)a^{k+1} \; A' \; (T-t_1) \; A \; (T-t_1)}{(a+A(T-t_1))} \left(1 - \frac{A(T-t_1)}{a+A\;(T-t_1)}\right)^{-(k+2)}\Big)$$

and

$$E(n^2) = \sum_{n=0}^{\infty} \frac{n^2(k+1)\;(k+2)\;\ldots\ldots\ldots(k+n)}{n!\;\;(a+A(t))^n} \; a^{k+1} \; A'(T-t_1)(A(T-t_1))^n$$

$$= a^{k+1} \; A' \; (T-t_1) \sum_{n=0}^{\infty} \frac{n(k+1)\ldots\ldots(k+n)}{(n-1)!\;\;(a+A(t))^n} \; (A(T-t_1))^n$$

$$= \quad a^{K+1} A' (T-t_1) \left(\frac{(k+1)(A(T-t_1)}{(a+A(T-t_1))} + \frac{2}{1!}(k+1)(k+2)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^2\right.$$

$$\left. + \frac{3}{2!}(k+1)(k+2)(k+3)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^3 + \ldots\ldots\right)$$

$$= \quad a^{k+1} A'(T-t_1)\frac{(k+1)A(T-t_1)}{(a+A(T-t_1))}\left(1+(k+2)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)\right.$$

$$+ \frac{(k+2)(k+3)}{1.2}\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^2 + \ldots\ldots$$

$$\left. + (k+2)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)\left(1+(k+3)\frac{A(T-t_1)}{a+A(T-t_1)} + \ldots\right)\right)$$

$$= \quad a^{k+1}\frac{A'(T-t_1)(k+1)A(T-t_1)}{(a+A(T-t_1)}\left(1-\frac{A(T-t_1)}{a+A(T-t_1)}\right)^{-(k+2)}$$

$$+ (k+2)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)\left(1-\frac{A(T-t_1)}{a+A(T-t_1)}\right)^{-(k+3)}$$

$$= \quad \frac{a^{k+1} A'(T-t_1)(k+1)A(T-t_1)}{(a+A(T-t_1))}\left(\frac{a}{a+A(T-t_1)}\right)$$

$$+ \quad a^{k+1} \quad \frac{A'(T-t_1)\ (k+1)\ (k+2)}{} \left(\frac{A\ (T-t_1)}{a+A(T-t_1)}\right)^2$$

$$\left(\frac{a}{a+A\ (T-t_1)}\right)^{-(k+3)}$$

$\Rightarrow$

$$\text{Var}\ (n) = E(n^2) - (E(n))^2$$

$$= \frac{a^{k+1}\ (k+1)\ A'(T-t_1)\ A\ (T-t_1)}{(a+A(T-t_1))} \left(\frac{a}{a+A\ (T-t_1)}\right)^{-(k+2)}$$

$$+ \ a^{k+1}\ A'\ (T-t_1)\ (k+1)(k+2) \left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)^2 \left(\frac{a}{a+A(T-t_1)}\right)^{-(k+3)}$$

$$- \left[(k+1)a^{k+1}\ A'(T-t_1)\left(\frac{A(T-t_1)}{a+A(T-t_1)}\right)\left(1-\frac{A(T-t_1)}{a+A(T-t_1)}\right)^{-(k+2)} \right]^2$$

Papers prepared from the contents of this thesis

1. On a Palm Probabilistic Approach for the Comparison of Cohort Fertility; Demography India, Vol.II,No.1, 1982, pp. 135-145.
2. On Probability for Obtaining Inter-Arrival Time Distribution in Weighted Poisson Process; Calcutta Statistical Association Bulletin, Vol. 32, Nos. 125-126, March and June, 1983, pp. 111-115.
3. On the application of Palm Probability for Obtaining Waiting Time Distribution Between the First and Higher Order Conceptions; Proceedings of the XIII International Conference on Stochastic Processes and their Applications. Demography India vo. 18, No. 2 page 277-312.
4. A Multistate Markov Chain Model for Evaluating a Sterilizationx Policy, Biometrical Journal vol. 29, No. 1, Page 57-64, 1984.
5. On a Martingale approach to a problem of Sterilization policy - Proceedings of the U.G.C. sponsored Seminar on 'Stochastic Modelling and Decision Making', University of Delhi, March 1986, Khanna Publishers, New Delhi, page 227-235.

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